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Multigrid-convergence of digital curvature estimators

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Multigrid-convergence of digital curvature estimators

Jacques-Olivier Lachaud

Abstract

Many methods have been proposed to estimate differential geometric quantities like curvature(s) on discrete data. A common characteristics is that they require (at least) one user-given scale or window parameter, which smoothes data to take care of both the sampling rate and possible perturbations. Digital shapes are specific discrete approximation of Euclidean shapes, which come from their digitization at a given grid step. They are thus subsets of the digital plane $\mathbb{Z}^d$. A digital geometric estimator is called *multigrid convergent* whenever the estimated quantity tends towards the expected geometric quantity as the grid step gets finer and finer. The problem is then: can we define curvature estimators that are multigrid convergent without such user-given parameter ? If so, what speed of convergence can we achieve ? We review here three digital curvature estimators that aim at this objective: a first one based on maximal digital circular arc, a second one using a global optimization procedure, a third one that is a digital counterpart to integral invariants and that works on 2D and 3D shapes. We close the exposition by a discussion about their respective properties and their ability to measure curvatures on gray-level images.

1. Introduction

Context and objectives. In many shape processing applications, the estimation of differential quantities on the shape boundary is usually an important step. Their correct estimation makes easier further processing, like quantitative evaluation, feature detection, shape matching or visualization. A considerable amount of approaches have been proposed to estimate curvature(s) given only discrete data. It is often desirable to have theoretical guarantees on the given estimation. This property is called *stability* in Geometry processing: given a continuous shape and a specific sampling of its boundary, the estimated measure should converge to the Euclidean one when the sampling become denser. Perhaps Amenta *et al.* [2] is one of the first work toward this goal.

When discrete data are **meshes**, most approaches are local and do not provide theoretical guarantees (see [41] and [18] for comprehensive evaluations, and Desbrun *et al.* [13] or Bobenko and Suris [3] for a more general theory). Results on the stability of curvature estimators are scarce. We may quote the result [42] for Gaussian curvature, integral curvature measures [10, 11], and to some extent integral invariants of [35, 34].

When discrete data are limited to **point clouds**, fitting polynomials is probably the most common approach (e.g. the osculating jets of Cazals and Pouget [5] is representative of these approaches), but the stability result is restricted to a perfect sampling. A more appealing family of techniques exploits the Voronoi diagram [1, 30, 31]. Several stability results are achieved even in presence of (Hausdorff noise), but they do not entail the stability of curvature estimations.

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Note that all the preceding approaches require some parameter tuning, the most important one determining the window of computation or the *scale* of the estimation in the terminology of scale-spaces.

This paper focuses on estimating the curvature (or curvature tensor) on the boundary of **digital shapes**. Such digital structures are subsets of the d-dimensional digital space $\mathbb{Z}^d$ and come generally from the digitization of some Euclidean shape. Of course, the curvature tensor estimation should be as close as possible to the curvature tensor of the underlying Euclidean shape before digitization. Digital data form a special case of discrete data with specific properties: (1) digital data cannot sample the boundary of the Euclidean shape (i.e. they do not lie on the shape boundary), (2) digital data are distributed around the true sample according to arithmetic noise, which looks rather uniform over a range $[-h, h]$ from a statistical point of view, where h is the digitization grid step. Another (weaker) way of stating these characteristics is to say that the Hausdorff distance between the Euclidean shape and its digitization is some $O(h)$. Of course, the quality of the estimation should be improved as the digitization step gets finer and finer. This property is called the *multigrid convergence* [22, 9]. It is thus similar in spirit with the *stability* property.

For 2D digital objects, a few approaches achieve multigrid convergence with some hypotheses. We quote the ones based either on binomial convolution principles [29, 15]. Algorithms are parametrized by the size of the support of the convolution kernel. Convergence theorem holds when such support size is an increasing function of the grid resolution and some shape characteristics. The polynomial fitting method of [36] is an almost parameter-free method for estimating second derivatives on functional digital data, and could perhaps be adapted to estimate the curvature along 2D contours. For 3D digital objects, several empirical methods exist for estimating curvatures, but none achieves multigrid convergence (e.g. see [27, 17]).

We look for digital curvature estimators with the following properties: (1) provably uniformly multigrid convergent, (2) accurate in practice, (3) computable in an exact manner, (4) efficiently evaluable at one point or everywhere, (5) robust to perturbations (i.e. bad digitization around the boundary, outliers), (6) parameter free. The last point is crucial since it allows the analysis of shapes *without any user supervision*. Note that parameter free convergence results have been obtained for length [40, 23] and normal vector estimation [24, 26, 12].

Paper organization. We review here three different approaches which aim at fulfilling these goals:

2D: Maximal Digital Circular Arcs (Section 3),

2D/3D: Constrained minimization of squared curvature (Section 4),

2D/3D: digital Integral Invariants (Section 5),

and we discuss there respective qualities in the last part. Note that the presentation of the different estimators may slightly differ from the original papers. Indeed, the intent is to homogenize notations and properties.

2. Notations and preliminaries

Shapes, digitization, boundary. In all subsequent sections, the symbol $\mathbb{X}$ denotes a family of compact simply connected subsets of $\mathbb{R}^d$ with continuous curvature fields. The *Gauss digitization* $\mathrm{Dig}_h(X)$ of $X \in \mathbb{X}$ with grid step h is defined as the set of integer points within the dilation of X by a factor $\frac{1}{h}$, i.e. $\mathrm{Dig}_h(X) \stackrel{def}{=} (\frac{1}{h} \cdot X) \cap \mathbb{Z}^d$. Any finite subset Z of $\mathbb{Z}^d$ is called a *digital shape*. Its *digital boundary* $\Delta(Z)$ is the set of $d-1$-dimensional cubical cells that form the topological border of $\cup_{z \in Z} \mathcal{Q}_z$, where $\mathcal{Q}_z$ is the unit cube centered on z. The *h-boundary* $\partial_h X$ of a shape X is the union of the cells of the digital boundary of $\mathrm{Dig}_h(X)$, rescaled by a factor h, i.e.

$$\partial_h X \stackrel{def}{=} h \cdot \cup_{c \in \Delta(\mathrm{Dig}_h(X))} c = \partial \cup_{z \in \mathrm{Dig}_h(X)} \mathcal{Q}_z.$$

Some of these notions are illustrated on Fig. 5.1, right.

Digital contour, multigrid convergence. Geometrically, the h-boundary of X is close (in the Hausdorff sense) to the topological boundary of X, but it is combinatorially equivalent to the digital boundary of the digitization of X with step h. In 2D, the digital boundary is often called *digital contour*, since it is easy to organize its 1-cells as one or several sequences of 1-cells (called *linels* or *cracks* depending on authors). Of course, a *pixel* is a point of $\mathbb{Z}^2$ and *voxel* is a point of $\mathbb{Z}^3$. The multigrid convergence property for local geometric estimators is formally defined below.

Definition 1. The estimator $\hat{\kappa}$ is *multigrid-convergent* toward the geometric quantity κ for the family $\mathbb{X}$ if and only if, for any $X \in \mathbb{X}$, there exits some positive h_0 such that, for any $0 < h < h_0$,

$$\forall x \in \partial X, \forall y \in \partial_h X \text{ with } \|y - x\|_1 \leq h, |\hat{\kappa}(\mathrm{Dig}_h(X), y, h) - \kappa(X, x)| \leq \tau_x(h),$$

where $\tau_{X,x} : \mathbb{R}^{+*} \to \mathbb{R}^+$ has null limit at 0. This function defines the speed of convergence of $\hat{\kappa}$ toward κ at point x of X. The convergence is *uniform* for X when every $\tau_{X,x}$ is bounded from above by a function τ_X independent of $x \in X$ with null limit at 0.

Medial axis, projection, and reach. For a compact set $X \subset \mathbb{R}^d$, let δ_X be the distance function to ∂X. The *medial axis* $MA_{\partial X}$ of ∂X is the subset of $\mathbb{R}^d$ whose points have at least two closest points on ∂X. Any point x of $\mathbb{R}^d \setminus \mathrm{MA}_{\partial X}$ has only one closest point on ∂X which we denote by $\xi_X(x)$. The mapping ξ_X is called *projection* and is defined for almost every point of $\mathbb{R}^d$. The *reach of* ∂X [16] is the infimum of $\{\delta_X(y), y \in \mathrm{MA}_{\partial X}\}$. It is denoted by $\rho_{\partial X}$. Note that any shape of $\mathbb{X}$ has a positive reach, which is related to the inverse of the maximal curvature but also to the gaps between shape parts.

3. Curvature by maximal digital circular arcs

The curvature estimator by maximal digital circular arcs (MDCA) was introduced in [38]. Maximal digital straight segments proved to be an excellent basis for tangent estimation. Hence maximal digital circular arcs is an excellent candidate for curvature estimation.

Let C be some digital contour to a digital shape Z. We look only at connected contours, since each connected component can be treated separately. In this case, the digital contour is a circular sequence of linels. Any proper connected part C' of C is a sequence of linels, whose *discrete length* is its number of linels. Each linel of C' lies between two edge-adjacent pixels, one pixel belonging to Z and called *interior* to C', the other pixel belonging to $\mathbb{Z}^2 \setminus Z$ and called *exterior* to C'.

Any part C' of C is a *digital circular arc* (DCA for short) if and only if the interior and exterior pixels of C' are circularly separable, i.e. there exists a (Euclidean) circle that either encloses the interior points without enclosing any exterior points or that encloses the exterior points without enclosing any interior points. Any map associating to a DCA A the value 0 if the interior and exterior points of A are linearly separable and the curvature of an arbitrary separating circle otherwise is denoted by k.

Any DCA A of C is *maximal* if and only if all the parts C' containing A, i.e. such that $A \subsetneq C' \subset C$, are not a DCA. The set of all maximal DCA (MDCA for short) that lie on a given contour is unique. Two distinct MDCA have two distinct starting linels and two distinct ending linels. The MDCA can be ordered according to the position of their first linel in the contour. Let us then denote by $(A_i)_{i \in \{1,\ldots,n\}}$ the sequence of the n MDCA lying on C.

As a result, a contour C can be partitioned without ambiguity into a sequence $(V_i)_{i \in \{1,\ldots,n\}}$ such that V_i is the set of linels closer to $m(A_i)$ than to any other linel $m(A_j)$, $j \in \{1, \ldots, n\}$ and $j \neq i$ (the first one with respect to the clockwise orientation of the contour is assumed to be closer in case of tie).

Definition 2. Let $Z \subset \mathbb{Z}^2$ be a digital shape of digital contour $C = \Delta(Z)$. Let p be any point of a linel $c \in C$. Then linel c and thus point p belongs to some V_i. The *parameter-free MDCA curvature estimator* $\hat{\kappa}^\star_{\mathrm{MDCA}}$ is defined as

$$\hat{\kappa}^\star_{\mathrm{MDCA}}(Z, p) \stackrel{def}{=} k(A_i). \tag{3.1}$$

The *rescaled MDCA curvature estimator* $\hat{\kappa}_{\mathrm{MDCA}}$ is naturally defined for some point $\hat{x} \in \partial_h X$ as

$$\hat{\kappa}_{\mathrm{MDCA}}(\mathrm{Dig}_h(X), \hat{x}, h) \stackrel{def}{=} \frac{1}{h} \hat{\kappa}^\star_{\mathrm{MDCA}}(\mathrm{Dig}_h(X), \frac{1}{h}\hat{x}). \tag{3.2}$$

This estimator approaches the curvature at a pointel as the curvature of the most-centered maximal digital circular arc around it (see Figure 3.1).

We have a limited multigrid convergence result for the MDCA estimator, whose validity depends on the asymptotic length of maximal digital circular arcs.

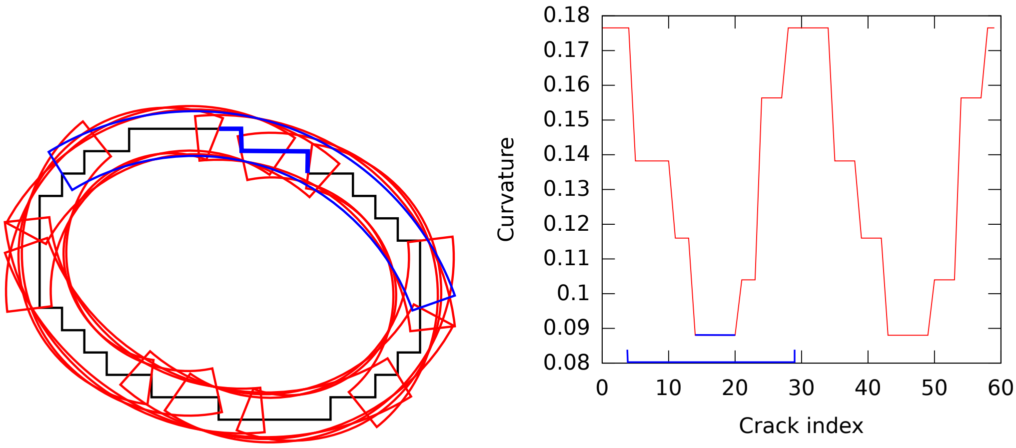


Figure 3.1: The set of MDCAs (12 arcs) is depicted in a) with pieces of rings along the contour of the digitization of an ellipse having a great axis of 9 pixels long and a small axis of 6 pixels long. The angle between the main orientation and the x-axis is equal to 1.9 radians. The curvature plot defined from the set of MDCAs is shown in b). The blue grid edges are those whose curvature depends on the radius of the blue MDCA.

Theorem 3 (Theorem 1, [38])**.** *Let $X \in \mathbb{X}$. If the Euclidean length of MDCAs along any $\partial_h X$ is lower bounded by $\Omega(h^a)$ and upper bounded by $O(h^b)$, $0 < b \le a < 1/2$, then the curvature estimator $\hat{\kappa}_{MDCA}$ is uniformly multigrid convergent to the curvature κ, with $\tau = O(h^{\min(1-2a,b)})$.*

Although experiments indicate that the length of MDCA falls into the hypothesis of this theorem, this fact is not proven. However, experiments show that this estimator is very accurate and convergent in practice.

4. Curvature by minimization of squared curvature

A completely different approach to curvature estimation was proposed in [19, 20]. Given a digital shape Z, the idea is to take into account *all* the smooth Euclidean shapes whose digitization is Z. Then, among all these shapes, the most representative shape for curvature estimation is the one that minimizes its total squared curvature. More precisely, the shape of reference to Z is sought in the "compactified" family

$$\mathbb{X}(Z) \stackrel{def}{=} \{X \in \mathbb{X}, \mathrm{Dig}_h(X \setminus \partial X) \subset Z \text{ and } \mathrm{Dig}_h(\mathbb{R}^d \setminus X) \subset \mathbb{Z}^d \setminus Z\}.$$

It just means that points exactly on the shape boundary may be either digitized "in" or "out". The shape of reference $X^{\mathrm{ref}}(Z)$ is the solution to the following minimization problem

$$X^{\mathrm{ref}}(Z) \stackrel{def}{=} \operatorname*{arg\,min}_{X \in \mathbb{X}(Z)} \int_{\partial X} \kappa^2 ds,$$

where κ is the mean curvature field over ∂X.

Definition 4. Let Z be a digital shape. Let p be any point inside some $d-1$-cell of $\Delta(Z)$. The *parameter-free MK2 curvature estimator* $\hat{\kappa}^{\star}_{\mathrm{MK}^2}$ is defined as

$$\hat{\kappa}^{\star}_{\mathrm{MK}^2}(Z, p) \stackrel{def}{=} \kappa(X^{\mathrm{ref}}(Z), \xi_{\partial X^{\mathrm{ref}}(Z)}(p)), \tag{4.1}$$

where $\xi_{\partial X^{\mathrm{ref}}(Z)}$ is the closest point to p on the boundary of the shape of reference to Z.

The *rescaled MK2 curvature estimator* $\hat{\kappa}_{\mathrm{MK}^2}$ is naturally defined for some point $\hat{x} \in \partial_h X$ as

$$\hat{\kappa}_{\mathrm{MK}^2}(\mathrm{Dig}_h(X), \hat{x}, h) \stackrel{def}{=} \frac{1}{h} \hat{\kappa}^{\star}_{\mathrm{MK}^2}(\mathrm{Dig}_h(X), \frac{1}{h}\hat{x}). \tag{4.2}$$

Finding this *shape of reference* is not a trivial task. In [20], a fast algorithm called *GMC* and using digital straight segments provides an approximation with no theoretical guarantees. This variational problem is also known as the minimization of a Willmore energy under constraints.

Hence, in [4] two other numerical techniques were proposed to find a solution: (i) a precise one based on convex optimization (but limited to convex shapes): (ii) a more versatile technique based on phase field approximation which is also extensible to 3D.

It is rather clear by definition that such an estimator should be uniformly multigrid convergent, provided one may determine the exact shape of reference. However, whatever the chosen algorithm or numerical technique, there is yet no theoretical guarantee on this estimator. A comprehensive 2D evaluation shows that it is experimentally multigrid convergent, although it is slightly less accurate than the MDCA estimator on perfect data. However, it is very stable and presents no oscillations in the result. It is thus easy to find the dominant points (maxima and minima of curvatures) and inflexion zones [21]. Another advantage of this approach is that it reconstructs a shape of reference. We have thus more than just an estimation of the curvature field. Figure 4.1 illustrates MK2 curvature estimations and contour reconstruction for 2D digital shapes, while Figure 4.2 gives 3D reconstruction results on a digital rabbit.

5. Curvature by digital integral invariants

Integral invariants were proposed in [35, 34] as a tool to analyze locally the geometry of triangulated mesh. The idea is to define integral quantities over the intersection of the shape X with a ball $B_r(x)$, centered on the point of interest x and of given radius r (see Fig. 5.1). These integral quantities are thus functions of the parameter r. For instance, the mean curvature is related to the volume of $X \cap B_r(x)$: it participates in the second term of the Taylor expansion of the volume at $r = 0$. We may note that a very similar tool was proposed earlier in [6].

It is possible to adapt this approach to digital data. In [7], the authors define a curvature estimator for 2D shapes and a mean curvature estimator for 3D shapes, based on digital integral invariants. The full curvature tensor is estimated by means of digital integral invariants in [8]. The 2D *parameter-free* curvature estimator is presented in [28].

Given a digital shape $Z \subset \mathbb{Z}^d$, the *discrete volume at step h* is defined as $\widehat{\mathrm{Vol}}_d(Z,h) \stackrel{def}{=} h^d \mathrm{Card}(Z)$.

Definition 5. Given a digital shape $Z \subset \mathbb{Z}^2$, any point $x \in \mathbb{R}^2$, some radius $r > 0$ and a gridstep $0 < h < r$, the *II curvature estimator* is defined as:

$$\hat{\kappa}_{r,\mathrm{II}}(Z,x,h) \stackrel{def}{=} \frac{3\pi}{2r} - \frac{3\widehat{\mathrm{Vol}}_2(B_{r/h}(\frac{1}{h}x) \cap Z, h)}{r^3}. \tag{5.1}$$

The 3D extension to this estimator, when $Z \subset \mathbb{Z}^3$ and $x \in \mathbb{R}^3$, is called the *II mean curvature estimator* and is written as:

$$\hat{\kappa}^m_{r,\mathrm{II}}(Z,x,h) \stackrel{def}{=} \frac{8}{3r} - \frac{4\widehat{\mathrm{Vol}}_3(B_{r/h}(\frac{1}{h}x) \cap Z, h)}{\pi r^4}. \tag{5.2}$$

When one wishes to estimate the full curvature tensor (principal curvatures, principal directions), we must estimate the second order moments of $X \cap B_r(x)$, also known as covariance matrix. For integers i, j, k, the *i,j,k-discrete moment of Z at step h* is defined as $\widehat{m}_{i,j,k}(Z,h) \stackrel{def}{=} h^{i+j+k} \sum_{(x,y,z)\in Z} x^i y^j z^k$. The *digital covariance matrix* is naturally defined as a centered version of the tensor of second order discrete moments:

$$\begin{aligned} \widehat{J}(Z,h) \stackrel{def}{=} & \begin{bmatrix} \widehat{m}_{2,0,0}(Z,h) & \widehat{m}_{1,1,0}(Z,h) & \widehat{m}_{1,0,1}(Z,h) \\ \widehat{m}_{1,1,0}(Z,h) & \widehat{m}_{0,2,0}(Z,h) & \widehat{m}_{0,1,1}(Z,h) \\ \widehat{m}_{1,0,1}(Z,h) & \widehat{m}_{0,1,1}(Z,h) & \widehat{m}_{0,0,2}(Z,h) \end{bmatrix} \\ & - \frac{1}{\widehat{m}_{0,0,0}(Z,h)} \begin{bmatrix} \widehat{m}_{1,0,0}(Z,h) \\ \widehat{m}_{0,1,0}(Z,h) \\ \widehat{m}_{0,0,1}(Z,h) \end{bmatrix} \otimes \begin{bmatrix} \widehat{m}_{1,0,0}(Z,h) \\ \widehat{m}_{0,1,0}(Z,h) \\ \widehat{m}_{0,0,1}(Z,h) \end{bmatrix}^T . \end{aligned}$$

Following the truncated Taylor expansion of [35], Theorem 2, we define estimators of curvatures from the diagonalization of the digital covariance matrix. Note that principal direction estimators are simply the two main eigenvectors of this matrix.

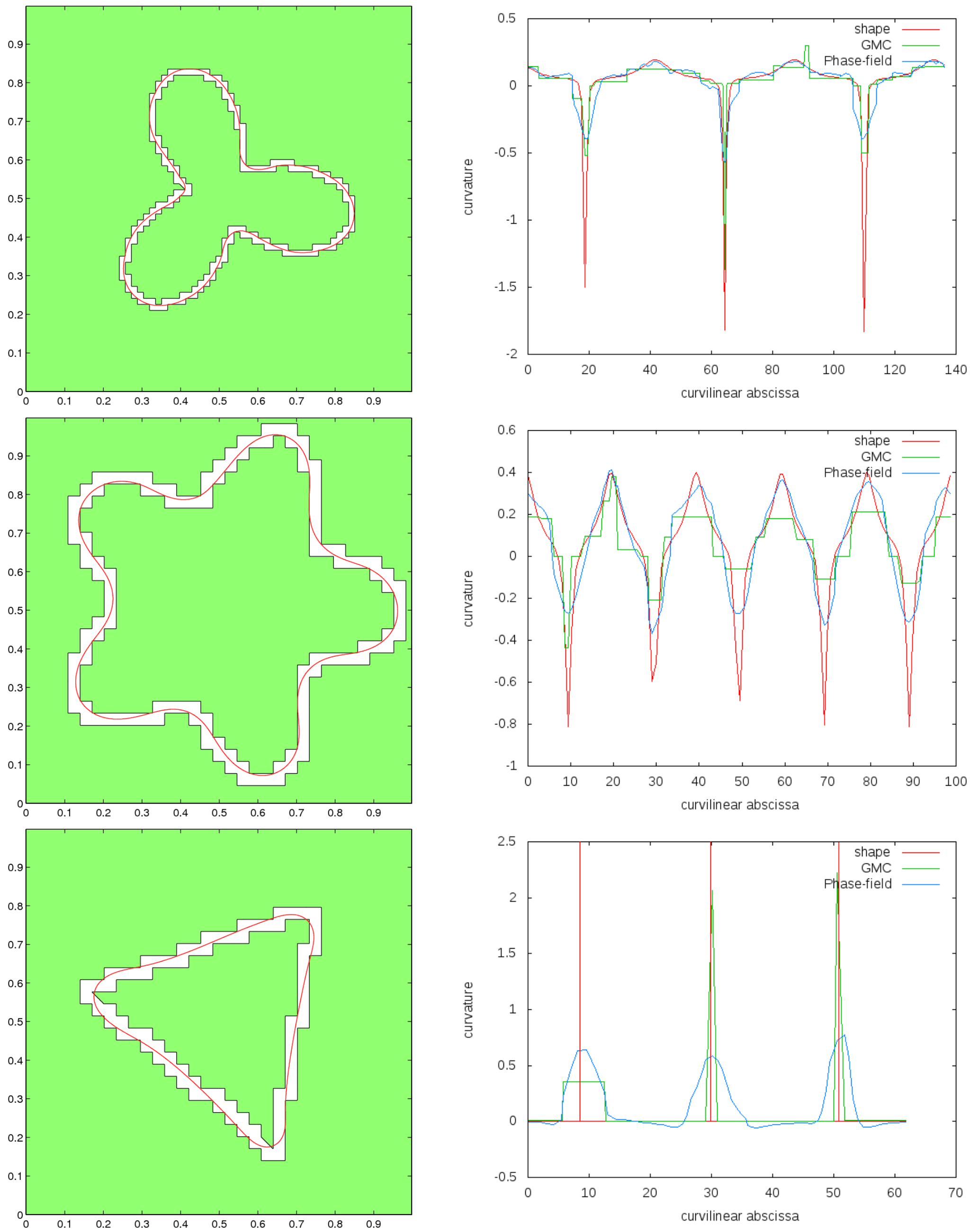


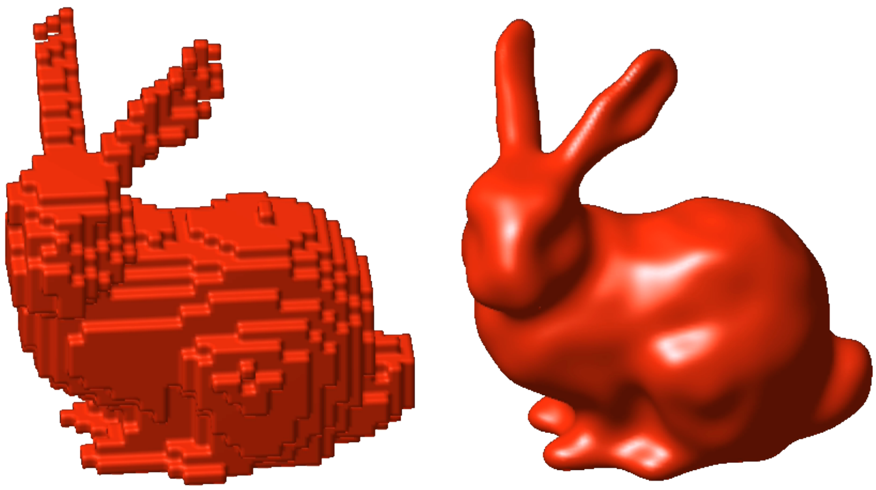
Figure 4.1: Digital shapes (left: contour as white digital path between interior and exterior pixels), shape of reference obtained with phase field reconstruction (left: red curve) and comparison of curvature estimations (right) with: true curvature (red), GMC algorithm (green) and Phase-field technique (cyan).

Definition 6. Given a digital shape $Z \subset \mathbb{Z}^3$, any point $x \in \mathbb{R}^3$, some radius $r > 0$ and a gridstep $0 < h < r$, the *II principal curvature estimators* are defined as:

$$\hat{\kappa}^1_{r,\mathrm{II}}(Z, x, h) = \frac{6}{\pi r^6}(\hat{\lambda}_2 - 3\hat{\lambda}_1) + \frac{8}{5r}, \quad (5.3)$$

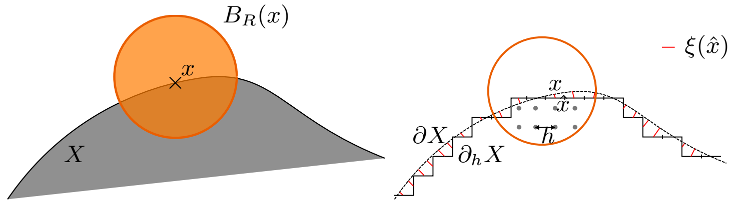
$$\hat{\kappa}^2_{r,\mathrm{II}}(Z, x, h) = \frac{6}{\pi r^6}(\hat{\lambda}_1 - 3\hat{\lambda}_2) + \frac{8}{5r}, \quad (5.4)$$

where $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the two greatest eigenvalues of $\widehat{J}(B_{r/h}(\frac{1}{h}x) \cap Z, h)$.



iteration 0 iteration 48

Figure 4.2: Phase field reconstruction of 3D digital rabbit.



(a) (b)

Figure 5.1: Integral invariant computation (*left*) and notations (*right*) in dimension 2.

The II curvature estimator can be made parameter-free. The idea is to use the average discrete length of all *maximal segments* of $\Delta(Z)$. Any part C' of a digital contour C is a *digital straight segment* (DSS for short) if and only if the interior and exterior pixels of C' are linearly separable, i.e. there exists a Euclidean straight line that separates interior points from exterior points. Any DSS M of C is a *maximal segment* if and only if all the parts M' containing M, i.e. such that $M \subsetneq M' \subset C$, are not a DSS.

When the digital shape is the digitization of some Euclidean shape X at gridstep h, the discrete length of maximal segments follows several asymptotic relations [12]. If we denote by $L_D(Z)$ the average of the discrete length of all maximal segments on the contour $\Delta(Z)$, then the precise fact used here is

$$\Theta(h^{-\frac{1}{3}}) \leq L_D(\mathrm{Dig}_h(X)) \leq \Theta(h^{-\frac{1}{3}} \log(h^{-1})).$$

Definition 7. Let $Z \subset \mathbb{Z}^2$ be a digital shape, and $C = \Delta(Z)$ its digital contour. Let p be any point of a linel of C. The *parameter-free II curvature estimator* $\hat{\kappa}^\star_{II}$ is defined as:

$$\hat{\kappa}^\star_{\mathrm{II}}(Z,p) \stackrel{def}{=} \frac{3\pi}{2\rho(Z)} - \frac{3A(Z,p)}{\rho(Z)^3} \tag{5.5}$$

where $\rho(Z) = (L_D(Z))^2$ and $A(Z,p) = \mathrm{Card}(B_{\rho(Z)}(p) \cap Z)$.

The *rescaled II curvature estimator* $\hat{\kappa}_{\mathrm{II}}$ is naturally defined for some point $\hat{x} \in \partial_h X$ as

$$\hat{\kappa}_{\mathrm{II}}(\mathrm{Dig}_h(X), \hat{x}, h) \stackrel{def}{=} \frac{1}{h} \hat{\kappa}^\star_{\mathrm{II}}(\mathrm{Dig}_h(X), \frac{1}{h}\hat{x}). \tag{5.6}$$

We do have several multigrid convergence for these estimators, for the family of shapes $\mathbb{X}$ (compact sets, C^3-smooth boundary):

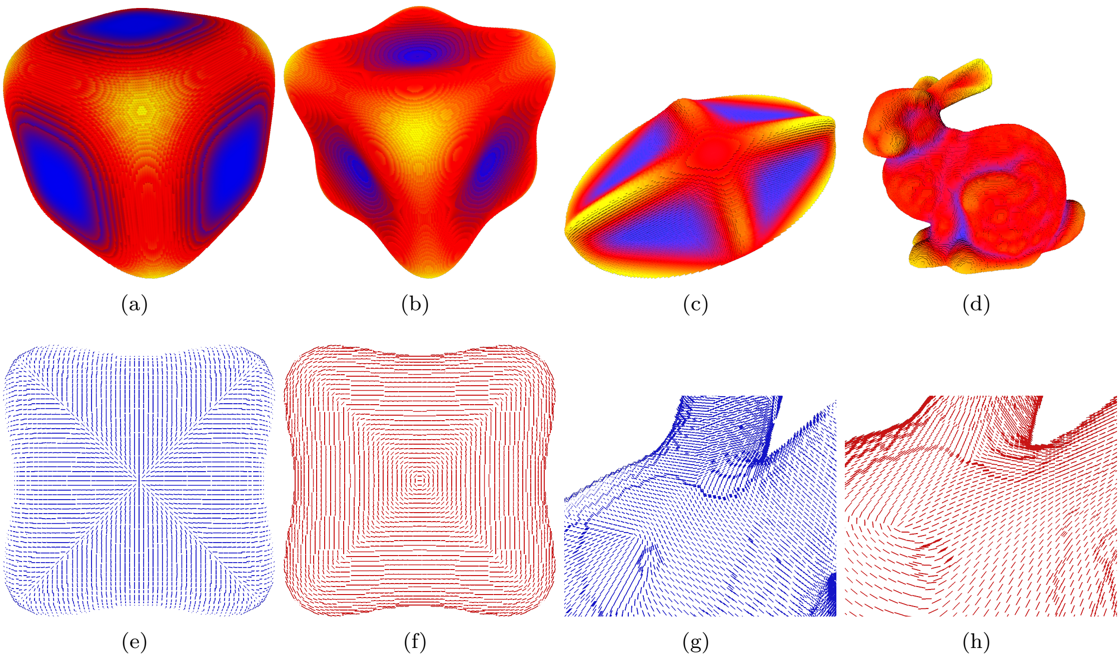


Figure 5.2: Illustration of 3D curvature estimation. Mean curvature on rounded cube (a), Goursat's surface (b), Leopold surface (c) and a bunny (d). First principal direction and second principal direction Goursat's surface (e and f) and Stanford bunny (g and h) .

estimator	quantity	parameters	convergence speed	reference
$\hat{\kappa}_{r,\mathrm{II}}$	2D curvature	$r = h^{\frac{1}{3}}$	$O(h^{\frac{1}{3}})$	[7]
$\hat{\kappa}^{m}_{r,\mathrm{II}}$	3D mean curvature	$r = h^{\frac{1}{3}}$	$O(h^{\frac{1}{3}})$	[7]
$\hat{\kappa}^{1}_{r,\mathrm{II}}$	1st principal curvature	$r = h^{\frac{1}{3}}$	$O(h^{\frac{1}{3}})$	[8]
$\hat{\kappa}^{2}_{r,\mathrm{II}}$	2nd principal curvature	$r = h^{\frac{1}{3}}$	$O(h^{\frac{1}{3}})$	[8]
$\hat{\kappa}^{\star}_{\mathrm{II}}$	2D unscaled curvature	parameter-free	$O(h^{\frac{1}{3}} \log^2(h^{-1}))$	[28]
$\hat{\kappa}_{\mathrm{II}}$	2D curvature	scale h	$O(h^{\frac{1}{3}} \log^2(h^{-1}))$	[28]

A very comprehensive set of experimental evaluation has been performed on II curvature estimators [7, 8, 28], as well as many comparisons with other approaches (MDCA, binomial convolutions [29, 15], jet fitting [5]). It is of course experimentally multigrid convergent. It is one of the most accurate in practice. Furthermore it is robust to noise due to its integral form. Figure 5.2 displays some results of estimators $\hat{\kappa}^{m}_{r,\mathrm{II}}$ and directions of $\hat{\kappa}^{1}_{r,\mathrm{II}}$ and $\hat{\kappa}^{2}_{r,\mathrm{II}}$. This has been implemented in the open-source library DGtal [14].

6. Discussion

We have presented three families of digital curvature estimators. It is clear that the gridstep h is necessary to get the correct *unit* for curvature, but we have shown above that we can define curvature estimators requiring *no parameter* if we assume simply that a pixel or voxel has unit length. Even better, we have exhibited one curvature estimator, the parameter-free II curvature estimator $\hat{\kappa}^{\star}_{\mathrm{II}}$, the multigrid convergence of which is established for the classical family $\mathbb{X}$ of compact shapes with C^3-boundary. Experiments show that this estimator competes with the accurate MDCA estimator but with the advantage of theoretical guarantees as well as a robustness to noise.

The following table summarizes the respective qualities of each curvature estimator, according to the desired properties described in the introduction. When an estimator meets fully a property, it is circled by a frame. The symbol n stands for the number of elements of $\Delta(Z)$. Note that $n = \Theta(h^{d-1})$ if d is the dimension of the space.

	convergence	accuracy	exact comp.	efficient	robust	parameters
2D estimators						
$\hat{\kappa}^{\star}_{\text{MDCA}}$	?	$O(h^{\frac{1}{3}})$	yes	$O(n^{\frac{4}{3}})$	no	unit
$\hat{\kappa}^{\star}_{\text{MK}^2}$ (GMC)	?	$\approx O(h^{\frac{1}{3}})$	Opt.	iter $\times O(n^{\frac{2}{3}})$	Hausdorff	$\approx$ unit
$\hat{\kappa}^{\star}_{\text{MK}^2}$ (PF)	?	$O(h^{\frac{1}{3}})$	Opt.	iter $\times O(n^2)$	no	$\approx$ unit
$\hat{\kappa}_{r,\text{II}}$	$O(h^{\frac{1}{3}})$	$O(h^{\frac{1}{3}})$	Yes	$O(n^{\frac{5}{3}})$	yes	need h
$\hat{\kappa}^{\star}_{\text{II}}$	$O(h^{\frac{1}{3}})$	$O(h^{\frac{1}{3}})$	Yes	$O(n^{\frac{5}{3}})$	yes	unit
3D estimators						
$\hat{\kappa}^{\star}_{\text{MK}^2}$ (PF)	?	?	Opt.	iter $\times O(n^{\frac{3}{2}})$	no	$\approx$ unit
$\hat{\kappa}^{m}_{r,\text{II}}$	$O(h^{\frac{1}{3}})$	$O(h^{\frac{1}{3}})$	Yes	$O(n^{\frac{5}{3}})$	yes	need h
$\hat{\kappa}^{1}_{r,\text{II}}$	$O(h^{\frac{1}{3}})$	$O(h^{\frac{1}{3}})$	Yes	$O(n^{\frac{5}{3}})$	yes	need h
$\hat{\kappa}^{2}_{r,\text{II}}$	$O(h^{\frac{1}{3}})$	$O(h^{\frac{1}{3}})$	Yes	$O(n^{\frac{5}{3}})$	yes	need h

It is clear that the next step is to define parameter-free 3D principal curvature estimators, with guaranteed multigrid convergence. For now, for this problem, only empirical solutions exist.

Relevance of digital estimators for estimating curvatures in gray-level images. Another natural question is the suitability of using digital curvature estimators on 2D or 3D gray-level image data. In this case, the input data is much richer than a simple binary image, since grey-level values could potentially be used for determining curvatures. Therefore there exists standard image derivation techniques to estimate the curvature of isocontours or isosurfaces within images, some involve derivative filters (e.g. [32]), specialized finite difference schemes [39, 33], or image structure tensor [37, 25].

Since it is parameter-free, we examine here the upwind finite difference scheme used in Level-Set (LS) techniques for estimating the mean curvature of some isosurface [39, 33], and we compare its accuracy to the 3D Integral Invariant (II) mean curvature estimator. Given a point p of value $I(p)$ in image I, LS curvature estimator uses grey-level information around p and estimates the mean curvature at p of the isosurface of value $I(p)$. On the other hand, II estimator processes only the **binary** image obtained by thresholding I at the value $I(p)$. We have compared numerically the respective performance of LS and II on a 3D image of a ball of radius 30, with a linear gradient of 50 at its boundary. We have also add a Gaussian noise of standard deviation $\sigma \in \{0, 1, 2, 3, 4, 5\}$, which is a very small perturbation considering that the background is 50 and the foreground is 200. Input data and experiments are illustrated on Figure 6.1. As one can see, even the noise with deviation 5 is almost imperceptible. If we look however at the average mean curvature computed by LS and II estimators, we see that their behaviors are dramatically different. LS estimator is accurate if the image is perfect (average is very close with 0.8% relative error but samples have a relative deviation of 38%). However, as soon a slight perturbation is added to the data, this estimator becomes very unstable. On the other hand, the accuracy of II estimator is related to a correct choice of ball radius (here $r \approx 10$ gives excellent results), but this estimator is stable whatever the noise. Note that the discussion above gives indication for the correct radius. Indeed it is as if we are digitizing a ball of radius 1 with gridstep $h = \frac{1}{30}$. A correct Euclidean radius for II estimator should follow $h^{\frac{1}{3}}$, hence the corresponding discrete radius is $\frac{h^{\frac{1}{3}}}{h} = h^{-\frac{2}{3}} \approx 9.65$.

To conclude, the image curvature estimator of [39, 33] is too unstable for analyzing real images coming from camera or biomedical devices. More robust techniques using image structure tensor [37, 25] can be parameterized to address noise in a global manner. The II digital curvature estimator gives reasonably accurate results even if the gray-level information is not reliable, and is stable with respect to noise (with zero mean). A natural open question is to extend digital curvature estimator to gray-level images (hence the shape is defined as a fuzzy characteristic set). II estimator may be a good candidate since the covariance matrix can be weighted accordingly.

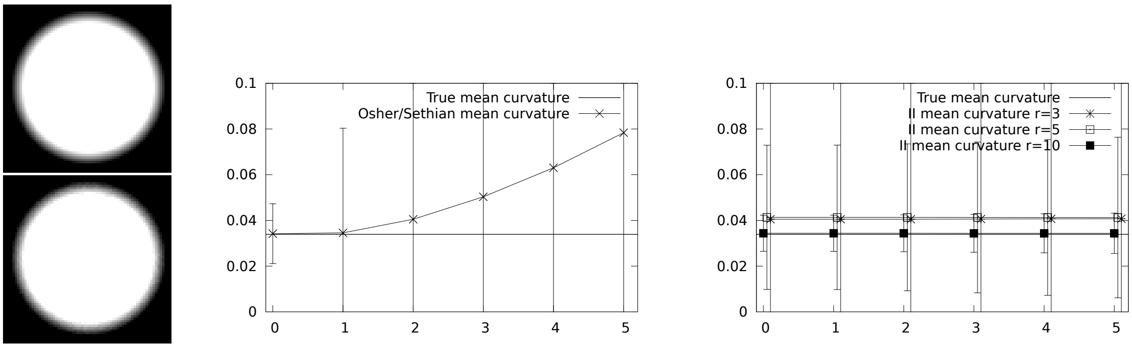


Figure 6.1: Mean curvature computation in a 3D gray-level image. **Left:** Slices in the input 3D 8-bit gray-level image, which represent a 3D ball of radius 30 with gradient 50 around its isosurface 128 (top image is perfect data, bottom image is data damaged with a Gaussian noise of deviation 5, i.e. PSNR=84.5). **Middle:** Average and deviation of mean curvature computed with LS estimator. **Right:** Average and deviation of mean curvature computed with II estimator, with several radii.

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