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Paul Baird

Curvature on a graph via its geometric spectrum

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Curvature on a graph via its geometric spectrum

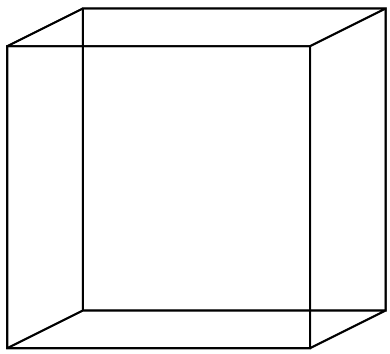
Paul Baird

Abstract

We approach the problem of defining curvature on a graph by attempting to attach a 'best-fit polytope' to each vertex, or more precisely what we refer to as a configured star. How this should be done depends upon the global structure of the graph which is reflected in its geometric spectrum. Mean curvature is the most natural curvature that arises in this context and corresponds to local liftings of the graph into a suitable Euclidean space. We discuss some examples.

1. Introduction

The problem we address is one of ascribing geometry to a graph using just its combinatorial structure. Geometry should *emerge* from the structure rather than being imposed upon it. Our approach is to appeal to the way we perceive objects in the world around us. A good starting point is what psychologists refer to as the Necker cube.



When we see this picture we generally perceive one of two possible 3-dimensional cubes. The reasons for this are a matter of cognitive science, but also the way the graph has been drawn on the piece of paper. With the graph so realized as a *framework* in 3-dimensional Euclidean space, we can begin to do geometry: edge length is defined; the Gauss curvature is defined at each vertex in terms of the angular deficit; other curvatures appear, such as the rotation of the Gauss map between adjacent vertices. But how has the realization of this graph come about?

To perform this procedure on an more general graph, we attempt to produce a "best-fit polytope" at each vertex, or rather, what we call a *configured star*, by lifting a vertex and its neighbours in a natural way into a Euclidean space $\mathbb{R}^N$. A configured star generalizes the star framework at each vertex of a regular polytope. More specifically, it consists of an internal vertex connected to n external vertices that have a particularly symmetric configuration (see below). In particular, any

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configured star has an axis which can be interpreted as the Gauss map at a vertex. The dimension N of the space into which we lift is a matter of choice but must always be less than or equal to the degree n of the vertex.

But now if we return to the cube pictured above, how do we decide whether to fit a cube to each vertex, or a regular tetrahedron, since both have vertices of degree 3 ? The answer is the cube because of the *global* combinatorial structure of the graph. In order to bring this global structure into play, we introduce the notion of *geometric spectrum* in [3, 1]. The geometric spectrum is a real-valued parameter γ defined on the vertices of a graph for which the quadratic difference equation :

$$\gamma(\Delta\phi)^2 = (\nabla\phi)^2 \tag{1.1}$$

has a non-trivial solution ϕ (see (2.2) below for the explicit form of this equation). For technical reasons that we explain below, we require that $\gamma < 1$. The smooth version of equation (1.1) applied to a hypersurface in Euclidean space shows that $1/\gamma = -H^2$ where H is the mean curvature. Since for a mesh which approximates a smooth hypersurface, equation (1.1) approximates its smooth counterpart, it is reasonable to consider γ as corresponding to mean curvature (or more precisely to $-1/H^2$). It is worth noting that for regular convex polyhedra, equation (1.1) is satisfied with γ positive in the case of a small number of vertices, for example the tetrahedron, and then becoming negative when the number of vertices increases, for example for the icosahedron, the dodecahedron and the 4-dimensional 600-cell, as the polyhedron approximates better a smooth round sphere.

For a complete graph on $N + 1$ vertices, the geometric spectrum consists of a single value $N/(N + 1)$ and the solution ϕ corresponds to the images of the vertices after an orthogonal projection of the regular N-simplex to the complex plane. At the other extreme, for a cyclic graph on n vertices, solutions ϕ to (1.1) correspond to realizations of the graph as an n-polygon in the plane with sides of equal length. Now the geometric spectrum has continuous components with complicated branching phenomena. An interpretation of the geometric spectrum as *information* implicit in a graph which may be exploited to enact structural change is discussed in the article [4].

Given a solution to (1.1) on a graph with $\gamma < 1$, it is shown in [5] how one has a local lifting property at each vertex: each vertex and its neighbours can be lifted to an invariant configured star in Euclidean space $\mathbb{R}^N$ with $2 \leq N \leq n$, where n is the degree of the vertex. In the case when $N = 3$, except for some special cases, this lifting is unique up to a one of two possibilities (for example, the two visualizations of the cube), as such, this dimension becomes the most interesting when degrees are ≥ 3. How we choose from the two possible liftings is then a global matter of correlating liftings at adjacent vertices (cf. the cube). For example, an Escher picture has local liftings with a global inconsistency.

Given a graph and a solution to (1.1), we can now make sense of distance and curvature. In general, provided we are only interested in relative distance between one part of the graph and another and curvature which doesn't depend on scale (for example Ricci curvature), these should depend only on the geometric spectrum γ and not on the solution ϕ.

2. The geometric spectrum

Any regular polytope in Euclidean space satisfies the following quadratic difference equation:

$$\frac{\gamma}{n}\Big(\sum_{y\sim x}\big(\phi(y)-\phi(x)\big)\Big)^2 = \sum_{y\sim x}\big(\phi(y)-\phi(x)\big)^2\,, \tag{2.1}$$

at each of its vertices x, where ϕ is an orthogonal projection to the complex plane, n is the (common) degree at each vertex (the number of edges incident with x) and $y \sim x$ means that y is connected to x by an edge. This fact is implicit in the work of Eastwood and Penrose [10] and made explicit by the author in [5]. The constant γ depends on the polytope and it can be either positive or negative but it is always < 1. For the convex regular polyhedra in $\mathbb{R}^3$ (the Platonic solids), the values of γ can be calculated by expressing the angular deficit at each vertex in terms of γ (cf. [1], Proposition 7.4) and then applying the classical theorem of Descartes: *for a convex polyhedron in $\mathbb{R}^3$, the sum over the vertices of their angular deficits is equal to 4π*. See table 2.1. Note that the value of γ in (2.1) is invariant by any similarity transformation of the polytope, which suggests that the intrinsic geometry of the polytope may be characterized by three aspects:

polyhedron	γ
tetrahedron	3/4
cube	0
octahedron	1/2
icosahedron	$\frac{2-\sqrt{5}}{3-\sqrt{5}} < 0$
dodecahedron	$\frac{3(1-\sqrt{5})}{2(3-\sqrt{5})} < 0$

Table 2.1: Values of γ for the Platonic solids.

(i) the fact that the underlying framework (or 1-skeleton) satisfies (2.1) ; (ii) the value of the constant γ ; (iii) the underlying combinatorial structure. Let us therefore proceed to generalize this construct to a more general graph.

Given a graph $\Gamma = (V, E)$ with vertex set V and edge set E, together with a real-valued function $\gamma : V \to \mathbb{R}$, we introduce the equation:

$$\frac{\gamma(x)}{n(x)}\Big(\sum_{y\sim x}\big(\phi(y)-\phi(x)\big)\Big)^2 = \sum_{y\sim x}\big(\phi(x)-\phi(y)\big)^2, \tag{2.2}$$

at each vertex x, where $\phi : V \to \mathbb{C}$ is a complex-valued function and $n(x)$ is the degree of Γ at x. Solutions with $\gamma \equiv 0$ have been called *holomorphic functions*[1] and have been used to give a description of massless fields in a combinatorial setting [6]. Note that the equations are invariant by the replacement of ϕ by the transformations

$$\phi \mapsto \lambda\phi + \mu \quad (\lambda, \mu \in \mathbb{C}), \quad \text{and} \quad \phi \mapsto \overline{\phi}. \tag{2.3}$$

We shall consider two solutions related in this way as *equivalent*.

If we set $\Delta\phi(x) = \frac{1}{n(x)}\sum_{y\sim x}\big(\phi(y)-\phi(x)\big)$ (the Laplacian) and $(\nabla\phi)^2(x) = \frac{1}{n(x)}\sum_{y\sim x}\big(\phi(y)-\phi(x)\big)^2$ (the symmetric square derivative), then equation (2.2) has the more economic expression (1.1) given in the Introduction. The Cauchy-Schwarz inequality shows that for a given vertex x if the values $\{\phi(y)-\phi(x) : y \sim x\}$ are real and not all zero, then $\gamma(x) \geq 1$ [5].

For a given graph, we would like to know what are the admissible functions $\gamma : V \to \mathbb{R}$ for which (2.2) has a solution. Define the *geometric spectrum* of Γ to be the collection of equivalence classes of functions:

$$\Sigma = \{\gamma : V \to [-\infty, 1) \subset \mathbb{R} : \exists \text{ non} - \text{const. } \phi : V \to \mathbb{C} \text{ satisfying (2.2) }\},$$

where two functions are identified when they determine a common solution ϕ and agree on the compliment of the set $\{x \in V : \Delta\phi(x) = (\nabla\phi)^2(x) = 0\}$. The upper bound on γ is a consequence of the Cauchy-Schwarz inequality and our requirement of invariance with respect to similarity transformations. We allow γ to take on the value $-\infty$ at points where the Laplacian vanishes.

By a *framework* in Euclidean space, we mean a graph that is realized as a subset of Euclidean space with edges straight line segments joining the vertices. We say that it is *immersed* if all vertices are distinct and *embedded* if it is immersed and edges only intersect at end points. The framework is called *invariant* if for a particular γ, it satisfies (2.2) with ϕ the restriction to the vertices of some orthogonal projection to the complex plane *independently of any similarity transformation of the framework*.

Questions that now arise are:

- For a given graph Γ, what is its geometric spectrum?
- Does a solution to (2.2) arise from an embedding of the graph as an invariant framework in Euclidean space?
- Even if the answer to the last question is no, can we still define geometric quantities such as edge length and curvature from a solution?
- To what extent do such quantities depend only on γ rather than on the choice of solution ϕ?

[1] A notion of *holomorphic function* somewhat similar to this has been introduced by S. Barré [7]; however, in addition to (2.2) with $\gamma \equiv 0$, Barré requires that ϕ be harmonic.

For an arbitrary graph, the geometric spectrum is determined by a fairly complicated set of algebraic equations. For graphs of sufficiently small order, these can be solved by the method of Gröbner bases with MAPLE, see [1]. The more connected a graph, the more restricted is its geometric spectrum as discussed in the Introduction.

In order to interpret γ, we consider equation (1.1) in the smooth case of a hypersurface in Euclidean space and find an interesting connection with mean-curvature.

Theorem 1. ([2]) *Let M^n be a smooth hypersurface in $\mathbb{R}^{n+1}$ $(n \geq 1)$ and let g denote the metric on M^n induced from the standard metric on $\mathbb{R}^n$. Let $\phi : (M^n, g) \to \mathbb{C}$ be any orthogonal projection; then*

$$(\Delta\phi)^2 = -H^2(\nabla\phi)^2\,, \tag{2.4}$$

where H is the mean curvature of M^n, and where in local coordinates, $\Delta\phi = g^{ij}(\phi_{ij} - \Gamma_{ij}^k\phi_k)$ and $(\nabla\phi)^2 = g^{ij}\phi_i\phi_j$ (summing over repeated indices).

In the case when $n = 1$, the theorem confirms the identity

$$c''(s) = \kappa(s) i c'(s)\,,$$

for a regular curve $c : I \subset \mathbb{R} \to \mathbb{C}$ parametrized with respect to arc length. It is necessary that M^n be a *hypersurface* in order to satisfy (1.1). For example, consider the surface in $\mathbb{R}^4$ parametrized in the form:

$$(x^1, x^2) \mapsto (x^1, x^2, x^1x^2, x^1 + x^2)\,.$$

Let $\phi : \mathbb{R}^4 \to \mathbb{C}$ be the projection $\phi(x^1, x^2, x^3, x^4) = x^1 + x^2 i$. Then is is readily checked that the function γ defined by (1.1) is not even real.

Given the above theorem, we expect an invariant framework that closely coincides with a smooth hypersurface to have γ approximately equal to $-1/H^2$ modulo a scaling factor (equation (2.4) is not scale invariant; in order to make it so, a volume term should be added).

3. Configured stars and the lifting problem

A star graph, or bipartite graph $K_{1,n}$, has one internal vertex connected to n external vertices; there are no other connections. A star framework in $\mathbb{R}^N$ with internal vertex located at the origin can be specified by a $(N \times n)$–matrix W whose columns are the components of the external vertices. We will refer to W as the *star matrix*. Provided the centre of mass of the external vertices does not coincide with the origin, then it defines a line through the origin which we refer to as the *axis of the star*. We are interested in a particular class of star frameworks whose external vertices form what we call a configuration in a plane orthogonal to the axis of the star.

A collection of points $\{\vec{v}_1, \ldots, \vec{v}_n\}$ in $\mathbb{R}^{N-1}$ forms a *configuration* if the $((N-1) \times n)$-matrix $U = (\vec{v}_1|\vec{v}_2|\cdots|\vec{v}_n)$ whose columns have as components the coordinates $v_{\ell j}$ of $\vec{v}_\ell$ $(j = 1, \ldots, N-1;\ \ell = 1, \ldots, n)$, satisfies:

$$UU^t = \rho I_{N-1}\,, \qquad \sum_{\ell=1}^n \vec{v}_\ell = \vec{0}\,, \tag{3.1}$$

for some non-zero constant ρ (necessarily positive), where $\vec{0}$ denotes the zero vector in $\mathbb{R}^{N-1}$ and U^t denotes the transpose of U. Necessarily, $\mathrm{rank}(U) = N - 1$ so that $n \geq N$. A star in $\mathbb{R}^N$ whose external vertices form a configuration in a plane not passing through the origin, is referred to as a *configured star*. An *invariant* of such a star is a quantity that is invariant by orthogonal transformation. The following lemma characterizes configured stars [5].

Lemma 2. *Consider a configured star in $\mathbb{R}^N$ $(N \geq 2)$ with internal vertex the origin connected to n external vertices $\{\vec{x}_1, \ldots, \vec{x}_n\}$ $(n \geq N)$. Let $W = (\vec{x}_1|\vec{x}_2|\cdots|\vec{x}_n)$ be the $(N \times n)$-matrix whose columns are the components $x_{\ell j}$ of $\vec{x}_\ell$ $(j = 1, \ldots, N;\ \ell = 1, \ldots, n)$. Then*

$$WW^t = \rho I_N + \sigma\vec{u}\vec{u}^t, \qquad \sum_{\ell=1}^n \vec{x}_\ell = \sqrt{n(\sigma+\rho)}\,\vec{u}\,, \tag{3.2}$$

where $\vec{u} \in \mathbb{R}^N$ is a unit vector called the axis of the star*, $\rho > 0$ and $\rho + \sigma > 0$. The quantities n, ρ, σ are all invariants of the star; the vector $\vec{u}$ is normal to the affine plane containing $\vec{x}_1, \ldots, \vec{x}_n$.*

Conversely, any matrix $W = (\vec{x}_1|\vec{x}_2|\cdots|\vec{x}_n)$ *satisfying* (3.2) *determines a configured star with internal vertex the origin and external vertices* $\vec{x}_1, \dots, \vec{x}_n$.

Corollary 3. *Let* $W = (\vec{x}_1|\vec{x}_2|\cdots|\vec{x}_n)$ *define a configured star and let* $\phi : \mathbb{R}^N \to \mathbb{C}$ *be orthogonal projection* $\phi(y_1, y_2, \dots, y_N) = y_1 + iy_2$. *Then if* $z_\ell = \phi(\vec{x}_\ell) = x_{\ell 1} + ix_{\ell 2}$, *we have*

$$\frac{\sigma}{n(\sigma+\rho)}\left(\sum_{\ell=1}^n z_\ell\right)^2 = \sum_{\ell=1}^n {z_\ell}^2\,,$$

where ρ *and* σ *are given by* (3.2). *In particular, with reference to equation* (2.2), $\gamma = \sigma/(\sigma+\rho)$ *is real and depends only on the star invariants.*

Proof. Let $\vec{u} = (u_1, \dots, u_N)$ be the unit normal to the plane of the star. Then for each $j = 1, \dots, N$, we have

$$\sum_{\ell=1}^n x_{\ell j} = \sqrt{n(\sigma+\rho)}\, u_j\,.$$

Thus

$$\begin{aligned}\left(\sum_{\ell=1}^n z_\ell\right)^2 &= \sum_{k,\ell=1}^n (x_{k1}x_{\ell 1} - x_{k2}x_{\ell 2} + 2ix_{k1}x_{\ell 2}) \\ &= n(\sigma+\rho)({u_1}^2 - {u_2}^2 + 2iu_1u_2) = n(\sigma+\rho)(u_1+iu_2)^2\,,\end{aligned}$$

whereas

$$\sum_{\ell=1}^n {z_\ell}^2 = \sum_{\ell=1}^n ({x_{\ell 1}}^2 - {x_{\ell 2}}^2 + 2ix_{\ell 1}x_{\ell 2}) = (WW^t)_{11} - (WW^t)_{22} + 2i(WW^t)_{12} = \sigma(u_1+iu_2)^2\,.$$

The formula now follows. □

To test whether a framework in Euclidean space is invariant, it suffices to see whether the star about each of its vertices is invariant at the internal vertex. A consequence of the above corollary is that any *configured* star is invariant at its internal vertex. The star framework about the vertex of any regular polytope is configured, so that the underlying framework of a regular polytope is invariant [5]. On the other hand, not all invariant stars are configured. For example, the star in $\mathbb{R}^3$ with $2r$ external vertices represented by the columns of the $(3 \times (2r))$ –matrix

$$W = \begin{pmatrix} x_1 & x_2 & \cdots & x_r & x_1 & x_2 & \cdots & x_r \\ s_1 & s_2 & \cdots & s_r & -s_1 & -s_2 & \cdots & -s_r \\ t_1 & t_2 & \cdots & t_r & -t_1 & -t_2 & \cdots & -t_r \end{pmatrix}\,,$$

where the vectors $\vec{s} = (s_1, \dots, s_r)$ and $\vec{t} = (t_1, \dots, t_r)$ are orthogonal and of the same length, is invariant, but it is only configured when $x_1 = x_2 = \cdots = x_r$. This kind of invariant star arises in the double cone construction of the next section.

Let ϕ be a solution to (2.2) and consider a particular vertex x. Let $y_1, \dots, y_n$ be the neighbours of x labelled in any order. Normalize the solution so that $\phi(x) = 0$ and then set $z_j = \phi(y_j)$. We suppose that not all z_j are zero. Write $z_j = \alpha_j + i\beta_j$ in real and imaginary parts. The lifting problem into $\mathbb{R}^3$ about the vertex x means finding an invariant star (whose internal vertex is located at the origin) with matrix

$$W = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & \beta_2 & \cdots & \beta_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

If we impose the further restriction that the invariant star be configured, then provided $\gamma(x) < 1$ and $\operatorname{rank} W = 3$, this can always be done with just a 2-fold ambiguity which corresponds to a choice of sign for the vector $\vec{x} := (x_1, \dots, x_n)$ [5, 3]. The 2-fold ambiguity is illustrated by the Necker cube discussed in the Introduction. In the case when $\operatorname{rank} W < 3$, then there is a 1-parameter family of solutions. This case occurs if and only if the complex numbers z_ℓ satisfy

$$n\sum_{\ell=1}^n |z_\ell|^2 + (\gamma-2)\Big|\sum_{\ell=1}^n z_\ell\Big|^2 = 0\,.$$

We may lift to an invariant configured star in $\mathbb{R}^N$ provided $N \leq n$, but for $N > 3$, there is generally a family of lifts [5].

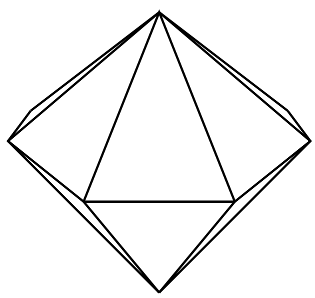
Thus we find, that apart from special cases, we have a lift about each vertex of a solution to (2.2) into $\mathbb{R}^3$ which is unique modulo translation along the axis of projection $\mathbb{R}^3 \to \mathbb{R}^2$ and up to the 2-fold ambiguity corresponding to the sign of $\vec{x}$. This already enables certain geometric quantities to be defined in an unambiguous way, for example edge length. Note that for a given edge we may have liftings at each of its end points which endow that edge with different lengths. In that case, we can take the average as a reasonable definition of edge length. Furthermore, the 2-fold ambiguity may sometimes be removed by a requirement of global consistency, as is the case with the cube: a choice at one vertex imposes a choice of lift at neighbouring vertices.

The problem of when a global lifting of a given graph exists remains relatively unexplored. An obvious geometric obstruction occurs when, as discussed in the above paragraph, the lifts of neighbouring vertices defines a *different* length to the connecting edge. This is particularly relevant when we try to lift into $\mathbb{R}^3$ since then, in general, edge length is unique (see, for example [5], Example 4.4). However, in general there is a smooth family of lifts into $\mathbb{R}^N$ when $N > 3$ subject to the constraint that $N \leq n$ ($n =$ degree of the vertex), so that it may still be possible to find a global lift into a higher dimension Euclidean space.

4. Examples

Given the interpretation of the parametre γ in terms of mean curvature, we now consider the problem of constructing invariant frameworks which have constant mean curvature. The underlying frameworks of regular polytopes provides examples, but are there others? The answer to this question is yes. Examples were given in [2], which we now outline.

If we take a regular n-sided polygon in the plane with vertices located at the points $e^{2k\pi i/n}$ ($k = 0, 1, \ldots, n-1$) and construct a double cone as illustrated below, then there is a unique height given by $\sin(2\pi/n)$ which makes this invariant, where by *height* we mean the distance from the plane of the polygon to one of the apexes.



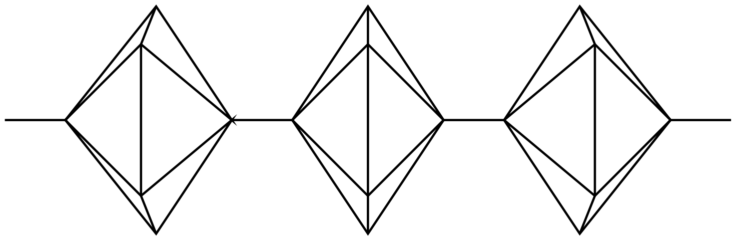
It is interesting to note that as $n \to \infty$ then the height approaches zero, so the form of the object approaches that of a disc. The value of γ at one of the lateral vertices is given by

$$\gamma_{\mathrm{lat}} = \frac{2(1 - 2\cos\frac{2\pi}{n} + 2\cos^2\frac{2\pi}{n})}{(2 - \cos\frac{2\pi}{n})^2}\,, \tag{4.1}$$

whereas at one of the apexes it is given by

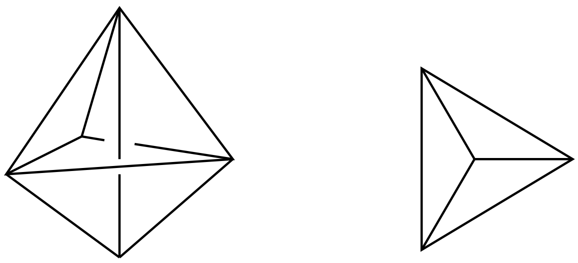
$$\gamma_{\mathrm{apex}} = \frac{2\sin^2\frac{2\pi}{n} - 1}{2\sin^2\frac{2\pi}{n}}\,.$$

Invariance at either of the apexes is a consequence of Lemma 2. On the other hand, invariance at one of the lateral vertices, that is one of the vertices of the planar polygon, now connected to the two apexes as well as to its two polygonal neighbours, is far less obvious. The double cone only has constant mean curvature when $n = 4$, in which case it corresponds to the octahedron. However, we can remedy this by attaching another double cone along the axis of the two apexes as illustrated below, where, for convenience, we draw the cone axis horizontally. In essence, we adjust the length of the edge joining the double cones until the parameter γ coincides with its value at one of the lateral vertices given by (4.1).



Such constant mean curvature frameworks are reminiscent of the period constant mean curvature surfaces of Delaunay [8].

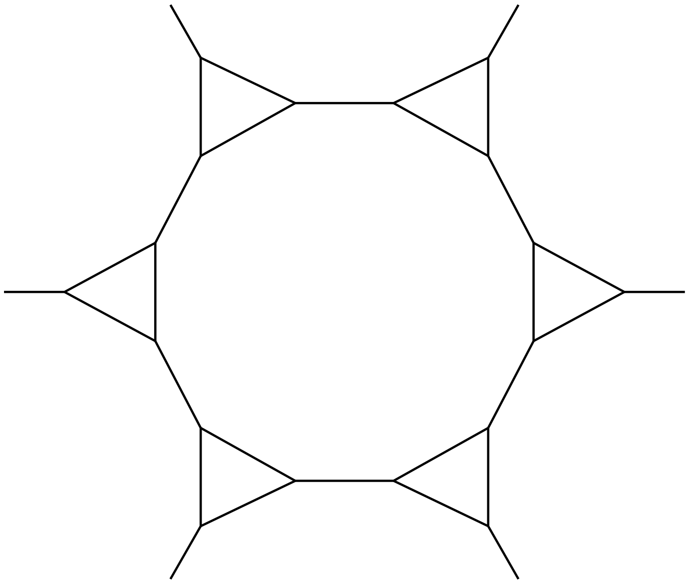
If we let x denote the distance along the axis at which we must attach the next cone in order that the new value of γ at an apex coincides with the lateral value of γ, then x is determined by a quadratic equation. In the case when $n = 3$, one solution is given by $x = -\sqrt{3}$ which is precisely the distance between the two apexes, so we obtain the constant mean curvature framework illustrated in the left-hand figure below, corresponding to the complete graph on 5 vertices. The value of γ is given by 4/5.



More generally, one can embed the complete graph on $N + 1$ vertices in an invariant way in $\mathbb{R}^{N-1}$ with γ constant equal to $N/(N+1)$. The case when $N = 3$, corresponding to an embedding of the complete graph on 4 vertices in the plane, is illustrated in the right-hand figure above. This example illustrates the need to allow equivalence classes of functions in our definition of the geometric spectrum. Here, both $\Delta\phi$ and $(\nabla\phi)^2$ vanish at the central vertex, so that γ is not well-defined by equation (2.2) at this vertex. However, we can assign to it the value it takes at the other vertices.

For $n = 4$ and $n = 5$, there are no real solutions for x (as we would expect for $n = 4$ since the octahedron is already of constant mean curvature and the attachment of another cone would destroy this). For $n = 6$, there is both a positive and negative root. If we take the negative value and perform a twist and proceed similarly for successive cones, then we obtain interlaced frameworks with quite complicated structure.

As a final construction, we can take a tiling of the plane, form double cones and stack layers one upon the other to obtain an analogue of triply periodic constant mean curvatures [1]. Such a tiling by dodecahedrons and triangles is illustrated below.



In this case, we form double cones over each triangle. Since the edges which connect each triangle (the edges of the dodecahedron) bisect the external angle, invariance is guaranteed. We then have to adjust the height of successive layers in order to arrange for γ to be constant.

The frameworks so constructed do not form the 1-skeleton of a polytope whose underlying topology is that of an immersed surface. We do not know if such examples exist other than the regular polytopes.

5. Other curvatures

By analogy with the smooth case, we view the function γ as related to mean curvature, at least when the framework is close to a smooth surface. But what about other curvatures? Various ideas were proposed in the arXiv article [1] which we now outline.

Suppose we have a solution to equation (2.2) and a well-defined lifting to a configured star at each vertex. For a given edge $e = \overline{xy}$ connecting vertices x and y, let $\theta(e)$ be the angle between the axes of the configured stars over x and y, respectively. Then this is well-defined and independent of the choice of representative solution under the equivalence (2.3). If we now let $\ell(e)$ denote the length of the edge $\overline{xy}$ as discussed at the end of Section 3, then the radius of the best-fit circle to that edge is $r(e) = \ell(e)/\theta(e)$. The reciprocal $k(e) := 1/r(e) = \theta(e)/\ell(e)$ may be taken as an analogue of *normal curvature*. This now depends on edge length and in particular on the choice of representative solution to (2.2). We can now make an alternative definition of mean curvature at a vertex x as the mean of the normal curvatures of edges incident with x. We do not know if after scaling is taken into account, there is a way to relate this to the function γ.

In smooth Riemannian geometry, the sectional curvature of a plane spanned by two vectors is the Gaussian curvature of a geodesic surface determined by the plane. In the discrete context, we can therefore define the sectional curvature associated to two edges e and f incident with a vertex x as $\mathrm{Sec}\,(e,f) := k(e)k(f)$.

Perhaps the most satisfying curvature in Riemannian geometry is the Ricci curvature which is scale invariant. For a given unit vector X, $\mathrm{Ric}\,(X,X)$ is the sum $\sum_j \mathrm{Sec}\,(X,Y_j)$ of the sectional curvatures of planes spanned by X and a set of orthonormal vectors $\{Y_j\}$ perpendicular to X. On a graph, for a given edge $e = \overline{xy}$ incident with a vertex x, we therefore define the Ricci curvature by:

$$\mathrm{Ric}_x(e,e) := \sum_{z\sim x, z\neq y} \theta(\overline{xy})\theta(\overline{xz})\,.$$

There seems to be no sensible way to define $\mathrm{Ric}\,(e,f)$ for distinct edges e and f incident with x. In the smooth case, this is usually achieved by the polarization identity, but there is no way to define the sum of two edges in the discrete context. Our definition of Ricci curvature is independent of the choice of representative solution to (2.2) under the equivalence (2.3). However, it is conceivable that for a given γ there may be two distinct classes of solutions, so it is not clear if the Ricci curvature depends only on γ.

Let us finally consider the Gaussian curvature. For a convex polyhedron in $\mathbb{R}^3$, the classical theorem of Descartes affirms that the sum of the angular deficits at each vertex is equal to 4π [9]. By angular deficit $\delta(x)$ at a vertex x, we mean 2π minus the sum of the internal angles at x of the faces which contain x. In our case, at each vertex, we have a lifting to a configured star, but we do not a priori have any underlying polytope. Thus in order to define the Gaussian curvature in terms of angular deficit, we require an ordering of the edges. One way to do this is to edge colour the graph, as discussed in [1]. However, we only expect to obtain approximate global theorems with our method for the following reason.

Suppose we have an invariant framework in $\mathbb{R}^3$ which projects to a solution of (2.2). When we perform a lift at each vertex to try to recover the original framework, we lift to a *configured* star, whereas the original star may not be configured. In essence, we sacrifice global lifting for unicity of lifting (up to 2-valuedness). As an example, consider the invariant double cone on the triangle discussed above. Now there is an underlying polytope and we can calculate angular deficit at each vertex in the traditional way. By the theorem of Descartes, the total angular deficit is 4π. However, let us calculate it by taking a lift to a configured star at each vertex as determined by the corresponding solution to (2.2). In the original figure, the stars at the lateral vertices are not configured, so an error will occur. We find:

$$\delta_{\text{apex}} = 2\pi - 3\arccos\frac{1}{7} \quad \text{and} \quad \delta_{\text{lat}} = 2\pi - 4\arccos\frac{5}{7},$$

to give a total curvature of

$$\delta_{\text{tot}} = 3\delta_{\text{lat}} + 2\delta_{\text{apex}} = 10\pi - 6\left(\arccos\frac{1}{7} + 2\arccos\frac{5}{7}\right) \sim 4.244 \times \pi\,.$$

References

[1] P. Baird. A class of quadratic difference equations on a finite graph. *arXiv:1109.3286 [math-ph].*

[2] P. Baird. Constant mean curvature polytopes and hypersurfaces via projections. *Differential Geom. and its Applications*, online version nov. 2013.

[3] P. Baird. Emergence of geometry in a combinatorial universe. *J. Geom. and Phys.*, **74**:185–195, (2013).

[4] P. Baird. Information, universality and consciousness: a relational perspective. *Mind and Matter*, **11**(2):21–43, (2013).

[5] P. Baird. An invariance property for frameworks in euclidean space. *Linear Algebra and its Applications*, **440**:243–265, (2014).

[6] P. Baird and M. Wehbe. Twistor theory on a finite graph. *Comm. Math. Phys.*, **304**(2):499–511, (2011).

[7] S. Barré. Real and discrete holomorphy: introduction to an algebraic approach. *J. Math. Pures Appl.*, **87**:495–513, (2007).

[8] C. Delaunay. Sur la surface de révolution dont la courbure moyenne est constante. **6**:309–320, (1841).

[9] R. Descartes. Progymnasmata de solidorum elementis. *Oeuvres de Descartes*, X:265–276.

[10] M. G. Eastwood and R. Penrose. Drawing with complex numbers. *Math. Intelligencer*, **22**:8–13, (2000).

Laboratoire de Mathématiques, de Bretagne Atlantique, Université de Bretagne Occidentale, 6 av. Victor Le Gorgeu – CS 93837, 29238 BREST CEDEX, FRANCE • paul.baird@univ-brest.fr