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A short survey on Gorenstein global dimension

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A short survey on Gorenstein global dimension

Driss Bennis

Abstract
This text gives a short overview of the recent works on Gorenstein global dimension of rings.

1. General results

Throughout this paper, if not specified otherwise, R denotes an associative ring with identity and all modules are unital left R-modules. Every result given for left modules has a corresponding one for right modules. For an R-module M, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$, and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective, injective, and flat dimension of M. We use $l.\mathrm{gldim}(R)$ and $\mathrm{wgldim}(R)$ to denote, respectively, the left global dimension and the weak global dimension of R. We assume the reader is familiar with Gorenstein homological algebra (see [10, 11, 12, 14] for a background). We use $\mathrm{Gpd}_R(M)$, $\mathrm{Gid}_R(M)$, and $\mathrm{Gfd}_R(M)$ to denote, respectively, the Gorenstein projective, injective, and flat dimension of M, and $l.\mathrm{Ggldim}(R)$ to denote the *left Gorenstein global dimension* of R (see its definition below after Theorem 1.1).

In the last few years, we have been mainly interested in the following Gorenstein global dimensions: $\sup\{\mathrm{Gpd}_R(M)\,|\,M \textit{ is an } R-\textit{module}\}$ and $\sup\{\mathrm{Gid}_R(M)\,|\,M \textit{ is an } R-\textit{module}\}$. The first objective was the following result, which is a Gorenstein analogue of a classical equality that is used to define the global dimension of rings.

Theorem 1.1. ([8, Theorem 1.1]) The following equality holds:

$$\sup\{\mathrm{Gpd}_R(M)\,|\,M \textit{ is an } R-\textit{module}\} = \sup\{\mathrm{Gid}_R(M)\,|\,M \textit{ is an } R-\textit{module}\}.$$

We call the common value of the quantities in the theorem the *left Gorenstein global dimension* of R and denoted by $l.\mathrm{Ggldim}(R)$. If R is commutative, then this quantity is simply called Gorenstein global dimension of R and denoted by $\mathrm{Ggldim}(R)$. The result above was proved earlier for left and right Noetherian rings (see [12, Theorem 12.3.1]). Namely, from [12, Theorem 12.3.1], we can deduce that, for a positive integer n, a left and right Noetherian ring R is n-Gorenstein if and only if $l.\mathrm{Ggldim}(R) \leq n$ if and only if $r.\mathrm{Ggldim}(R) \leq n$. Recall that a ring R is called n-Gorenstein if it is both left and right Noetherian with self-injective dimension at most n on both left and right sides (see [12, Definitions 9.1.1 and 9.1.9]).

As in the case of modules, the Gorenstein global dimension is a refinement of the classical global dimension. Precisely, for any ring R, $l.\mathrm{Ggldim}(R) \leq l.\mathrm{gldim}(R)$, and equality holds if $\mathrm{wgldim}(R) < \infty$ [8, Corollary 1.2].

Gorentein global dimension can be characterized as follows:

Theorem 1.2. ([8, Lemma 2.1] and [17, Theorem 2.1]) For a ring R and a positive integer n, the following assertions are equivalent:

1. $l.\mathrm{Ggldim}(R) \leq n$;
2. $l.\mathrm{Ggldim}(R) < \infty$, and $\mathrm{id}_R(P) \leq n$ for every projective left R-module P (or equivalently, for every left R-module P with finite projective dimension);

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(3) $l.\mathrm{Ggldim}(R) < \infty$, and $\mathrm{pd}_R(I) \leq n$ for every injective left R-module I (or equivalently, for every left R-module I with finite injective dimension);
(4) $\mathrm{id}_R(P) \leq n$ for every projective left R-module P, and $\mathrm{pd}_R(I) \leq n$ for every injective left R-module I.
(5) $\mathrm{id}_R(P) \leq n$ for every left R-module P with finite projective dimension, and $\mathrm{pd}_R(I) \leq n$ for every left R-module I with finite injective dimension.

Proof. Each implication, except $4 \Rightarrow 1$, can be easily deduced from the paper [8]. Here, using the notion of (n, m)-SG-projective modules [1], we give a short new proof of the implication $4 \Rightarrow 1$, which is established in [17]. Recall that, for integers $n \geq 1$ and $m \geq 0$, a module M is called (n, m)-SG-projective if there exists an exact sequence of modules, $0 \to M \to Q_n \to \cdots \to Q_1 \to M \to 0$, where $\mathrm{pd}(Q_i) \leq m$ for $1 \leq i \leq n$, and further, $\mathrm{Ext}^i(M, Q) = 0$ for any $i > m$ and for any projective module Q.
$4 \Rightarrow 1$. Let M be an R-module. Using both projective and injective resolutions of M we get an exact complex of the form $\cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$, where P_i are projective for $i \geq 0$ and injective for $i < 0$, such that $M \cong \mathrm{Im}(P_0 \to P_{-1})$. Now similarly to the proof of [4, Theorem 2.7] we can show that M is a direct summand of an R-module N occurring in an exact sequence of the form $0 \to N \to H \to N \to 0$ where $H = \oplus_i P_i$. Then using the condition on projective and injective modules we can see easily that N is a $(1, n)$-SG-projective R-module, and hence, that $\mathrm{Gpd}(N) \leq n$ [1, Theorem 2.4]. This implies that $\mathrm{Gpd}(M) \leq n$, as desired. □

The study of rings with small Gorenstein global dimension was begun in [9] for commutative rings of Gorenstein global dimension 0. In general, we have:

Proposition 1.3. ([8, Proposition 2.6]) The following are equivalent:
(1) R is quasi-Frobenius (i.e. left and right Noetherian and both left and right self-injective);
(2) $l.\mathrm{Ggldim}(R) = 0$;
(3) $r.\mathrm{Ggldim}(R) = 0$.

A characterization of rings with Gorenstein global dimension at most 1 is still an open problem, though it is settled in [3] for commutative integral domains as follows.

Theorem 1.4. ([3, Theorem 3.4]) Let R be a commutative integral domain, then $l.\mathrm{Ggldim}(R) \leq 1$ if and only if R is 1-Gorenstein.

A particular case of rings with finite Gorentein global dimension is introduced and studied in [2].

2. Gorentein global dimension of some classical ring constructions

Just as in the classical case, the Gorenstein global dimension of some known ring constructions have been of interest. The first work in this subject was [7], where Gorenstein global dimension of polynomial rings was computed.

Theorem 2.1. ([7, Theorems 2.1 and 3.1]) For any polynomial ring $R[X_1, X_2, ..., X_n]$ in n indeterminates over a commutative ring R, we have

$$\mathrm{Ggldim}(R[X_1, X_2, ..., X_n]) = \mathrm{Ggldim}(R) + n.$$

Recall that a commutative square of ring homomorphisms

$$(\square) \qquad \begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ R_2 & \underset{j_2}{\twoheadrightarrow} & R' \end{array}$$

is said to be a pullback square, if given $r_1 \in R_1$ and $r_2 \in R_2$ with $j_1(r_1) = j_2(r_2)$ there exists a unique element $r \in R$ such that $i_1(r) = r_1$ and $i_2(r) = r_2$. The ring R is called a pullback of R_1 and R_2 over R'. In what follows, we assume that j_2 is surjective.

The study of global dimensions of pullback rings has been the subject of several works and has led to interesting examples. Some of these examples have been used to solve important conjectures. Therefore, it is natural to try to establish analogous results for Gorenstein dimensions. In particular, we have been interested in finding analogues of the following result of Kirkman and Kuzmanovich [15, Theorem 2]: $l.\mathrm{gldim}(R) \leq \max_i\{l.\mathrm{gldim}(R_i) + \mathrm{fd}(R_i)_R\}$. In the paper [3], we attempted to extend this result to the setting of Gorenstein dimensions as follows: $l.\mathrm{Ggldim}(R) \leq \max_i\{l.\mathrm{Ggldim}(R_i) + \mathrm{Gfd}(R_i)_R\}$. This question is still open. However, we established the following partial answers.

Theorem 2.2. ([7, Theorems 2.1, 2.3 and 3.1]) Consider a pullback diagram of type ($\square$), with j_2 surjective.

(1) If $l.\mathrm{Ggldim}(R) < \infty$, then

$$l.\mathrm{Ggldim}(R) \leq \max_i\{l.\mathrm{Ggldim}(R_i) + \mathrm{Gfd}(R_i)_R\}.$$

(2) If the rings of ($\square$) are commutative and R is coherent, then

$$\mathrm{Ggldim}(R) \leq \max_i\{\mathrm{gldim}(R_i) + \mathrm{Gpd}_R(R_i)\}.$$

(3) If the rings of ($\square$) are commutative and R is Noetherian, then

$$\mathrm{Ggldim}(R) \leq \max_i\{\mathrm{gldim}(R_i) + \mathrm{Gfd}_R(R_i)\}.$$

Also trivial ring extensions are of interest. Recall that the trivial extension of a ring R by an R-module M is the ring denoted by $R \ltimes M$ whose underling group is $A \times M$ with multiplication given by $(r, m)(r', m') = (rr', rm' + r'm)$. From [5, Proposition 2.5], we have $\mathrm{Ggldim}(R \ltimes R) = \mathrm{Ggldim}(R)$ for any coherent commutative ring R. Also [16, Theorem 3.1] shows that, for a semi-local commutative ring (R, m) with maximal ideal m and an R-module E with $mE = 0$, we have $\mathrm{Ggldim}(R \ltimes E) = \infty$ if R is Noetherian and not field and E is finitely generated (i.e., $R \ltimes E$ is Noetherian). Also we have either $\mathrm{Ggldim}(R \ltimes E) = \infty$ or $\mathrm{Ggldim}(R \ltimes E) = 0$ if R is perfect.

Finally, we note that in addition to the open questions mentioned above, several natural others ones are of interest. For instance, the study of Gorenstein global dimension of tensor product of rings might be used to solve some old problems, such as when the tensor product of Gorenstein rings is Gorenstein (see [13]).

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