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# Semigroup-theoretical characterizations of arithmetical invariants with applications to numerical monoids and Krull monoids

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**Abstract**

Arithmetical invariants—such as sets of lengths, catenary and tame degrees—describe the non-uniqueness of factorizations in atomic monoids.We study these arithmetical invariants by the monoid of relations and by presentations of the involved monoids. The abstract results will be applied to numerical monoids and to Krull monoids.

## 1. Introduction

This is an extended abstract of the paper [2]. Its main results were presented in a talk of the second author at the *Third International Meeting on Integer Valued Polynomials and Problems in Commutative Algebra, December 2010, Marseilles.* We thank the organizers for the kind invitation.

Factorization theory describes the non-uniqueness of factorizations into irreducible elements of atomic monoids by arithmetical invariants, and it studies the relationship between these arithmetical invariants and algebraic invariants of the objects under consideration. In abstract semigroup theory, minimal relations and presentations are key tools to describe the algebraic structure of semigroups. Thus, there should be natural connections between the arithmetical invariants of factorization theory and the presentations of the semigroup. It were Scott T. Chapman and the second author—together with various coauthors—who made first steps to unveil these connections and to apply them successfully for further investigations (see [4, 3]). In the last years there has been a series of papers in this direction. Results have been carried over from finitely generated monoids to atomic monoids, and they have been extended to a larger class of invariants (see [8, 15, 13, 14]). Moreover, there was also progress from the computational point of view. The abstract characterizations in terms of relations gave rise to the development of explicit algorithms which partly have been implemented in GAP (see [5]).

In this extended abstract we sketch two results from [2]. The first one deals with the catenary degree and the tame degree, and the second is about unions of sets of lengths.

By a *monoid*, we mean a commutative cancellative semigroup with unit element. Let $S$ be a monoid, $a \in S$ and $a = u_1 \cdot \ldots \cdot u_k$ a factorization of $a$ into atoms $u_1, \ldots, u_k$. Then $k$ is called the length of the

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factorization, and $\mathsf{L}(a) = \{k \mid a \text{ has a factorization of length } k\} \subset \mathbb{N}$ is the set of length of $a$. For units $a \in S^\times$, we set $\mathsf{L}(a) = \{0\}$. We say that $S$ is a BF-monoid if $\mathsf{L}(a)$ is finite and non-empty for all $a \in S$.

*Throughout, let $S$ be a* BF*-monoid.*

## 2. Arithmetical invariants and presentations of monoids

We denote by $\mathcal{A}(S)$ the set of atoms (irreducible elements) of $S$, by $S^\times$ the group of invertible elements, and by $S_{\mathrm{red}} = \{aS^\times \mid a \in S\}$ the associated reduced monoid of $S$. The free (abelian) monoid $\mathsf{Z}(S)$ with basis $\mathcal{A}(S_{\mathrm{red}})$ is called the *factorization monoid* of $S$, the unique homomorphism

$$\pi\colon \mathsf{Z}(S) \to S_{\mathrm{red}} \quad \text{satisfying} \quad \pi(u) = u \quad \text{for each} \quad u \in \mathcal{A}(S_{\mathrm{red}})$$

is called the *factorization homomorphism* of $S$ and

$$\sim_S = \{(x, y) \in \mathsf{Z}(S) \times \mathsf{Z}(S) \mid \pi(x) = \pi(y)\}$$

the *monoid of relations* of $S$. Clearly, $\sim_S \subset \mathsf{Z}(S) \times \mathsf{Z}(S)$ is saturated and hence $\sim_S$ is a Krull monoid. Let $\sigma \subset \sim_S$ be a subset. Then $\sigma^{-1} = \{(x, y) \in \sigma \mid (y, x) \in \sigma\}$, and $\sigma$ is called a *presentation* of $S$ if the congruence generated by $\sigma$ equals $\sim_S$ (equivalently, if $(x, y) \in \mathsf{Z}(S) \times \mathsf{Z}(S)$, then $(x, y) \in \sim_S$ if and only if there exist $z_0, \ldots, z_k \in \mathsf{Z}(S)$ such that $x = z_0$, $z_k = y$, and, for all $i \in [1, k]$, $(z_{i-1}, z_i) = (x_{i-1}w_i, x_i w_i)$ with $w_i \in \mathsf{Z}(S)$ and $(x_{i-1}, x_i) \in \sigma \cup \sigma^{-1}$). A presentation $\sigma$ is said to be

- *minimal* if no proper subset of $\sigma$ generates $\sim_S$ (see [16, Chapter 9] for characterizations of minimal presentations in our setting).
- *generic* if $\sigma$ is minimal and for all $(x, y) \in \sigma$ we have $\mathrm{supp}(xy) = \mathcal{A}(S_{\mathrm{red}})$.

If $S$ has a generic presentation, then $S_{\mathrm{red}}$ is finitely generated and has no primes.

For $a \in S$, the set $\mathsf{Z}(a) = \pi^{-1}(aS^\times)$ is the set of factorizations of $a$. For $z, z' \in \mathsf{Z}(S)$, we can write

$$z = u_1 \cdot \ldots \cdot u_l v_1 \cdot \ldots \cdot v_m \quad \text{and} \quad z' = u_1 \cdot \ldots \cdot u_l w_1 \cdot \ldots \cdot w_n\,,$$

where $l, m, n \in \mathbb{N}_0$ and $u_1, \ldots, u_l, v_1, \ldots, v_m, w_1, \ldots, w_n \in \mathcal{A}(S_{\mathrm{red}})$ are such that

$$\{v_1, \ldots, v_m\} \cap \{w_1, \ldots, w_n\} = \emptyset\,.$$

Then $\mathsf{d}(z, z') = \max\{m, n\} \in \mathbb{N}_0$ is the *distance* between $z$ and $z'$. For subsets $X, Y \subset \mathsf{Z}(S)$, we set $\mathsf{d}(X, Y) = \min\{\mathsf{d}(x, y) \mid x \in X, y \in Y\} \in \mathbb{N}_0$.

For convenience we repeat the definition of the catenary degree $\mathsf{c}(S)$ and of the adjacent catenary degree $\mathsf{c}_{\mathrm{adj}}(S)$. For the definition of the tame degree $\mathsf{t}(S)$ we refer to [2]. For $a \in S$, let $\mathsf{c}(a) \in \mathbb{N}_0$ denote the smallest $N \in \mathbb{N}_0$ with the following property: for all $z, z' \in \mathsf{Z}(a)$, there exist $z_0 = z, z_1, \ldots, z_k = z' \in \mathsf{Z}(a)$ such that $\mathsf{d}(z_{i-1}, z_i) \le N$ for all $i \in [1, k]$. For $k \in \mathbb{Z}$, let $\mathsf{Z}_k(a) = \{z \in \mathsf{Z}(a) \mid |z| = k\}$ denote the set of factorizations of $a$ having length $k$, and define

$$\mathsf{c}_{\mathrm{adj}}(a) = \sup\{\mathsf{d}\big(\mathsf{Z}_k(a), \mathsf{Z}_l(a)\big) \mid k, l \in \mathsf{L}(a) \text{ are adjacent}\}\,.$$

Then

$$\mathsf{c}(S) = \sup\{\mathsf{c}(b) \mid b \in S\} \in \mathbb{N}_0 \cup \{\infty\} \quad \text{resp.} \quad \mathsf{c}_{\mathrm{adj}}(S) = \sup\{\mathsf{c}_{\mathrm{adj}}(b) \mid b \in S\} \in \mathbb{N}_0 \cup \{\infty\}$$

are the *catenary degree* of $S$, resp. the *adjacent catenary degree* of $S$.

There is the basic inequality $\mathsf{c}(S) \le \mathsf{t}(S)$, showing that the catenary degree is bounded above by the tame degree. The following result offers a sufficient condition to enforce equality. Its proof is based on a new characterization of the $\omega$-invariant $\omega(S)$, for which we have $\mathsf{c}(S) \le \omega(S) \le \mathsf{t}(S)$, and it allows applications to the monoid of $v$-invertible $v$-ideals in a weakly Krull domain (see [2, Theorem 5.6 and Corollary 5.7]).

**Theorem 2.1.** *Let $S$ be atomic, $P \subset S$ a set of representatives of the set of primes of $S$ and $T$ the set of all $a \in S$ such that $p \nmid a$ for all $p \in P$. Suppose that $T = \coprod_{i \in I} T_i$, $T \neq T^\times$ and that there is an $i^* \in I$ such that $T_{i^*}$ has a generic presentation and $\mathsf{t}(T_{i^*}) = \mathsf{t}(T)$. Then $\mathsf{c}(S) = \omega(S) = \mathsf{t}(S)$.*

The finiteness of the tame degree is a strong property enforcing the finiteness of the elasticity, of the $\omega$-invariant and of the catenary degree (for a large class of noetherian domains which are tame, see [12]). However, its relationship with the adjacent catenary degree is still open. Theorem 5.1 in [11] states that in a tame monoid $S$ there is a constant $M \in \mathbb{N}$ with the following property:

For each two adjacent lengths $k, l \in \mathsf{L}(a) \cap [\min \mathsf{L}(a) + M, \max \mathsf{L}(a) - M]$ we have $\mathsf{d}\big(\mathsf{Z}_k(a), \mathsf{Z}_l(a)\big) \le M$.

This result is an indication that the following problem could have a positive answer.

**Problem 2.2.** Does a monoid $S$ with finite tame degree $\mathsf{t}(S)$ have finite adjacent catenary degree $\mathsf{c}_{\mathrm{adj}}(S)$?

## 3. Unions of sets of lengths

The structure of sets of lengths and the structure of their unions are a central topic in the theory of non-unique factorizations (see [10, Section 4.7] for an overview, and [17, 11] for some recent progress). Let $k \in \mathbb{N}$ and suppose that $S \neq S^\times$. Then

$$\mathcal{V}_k(S) \;=\; \bigcup_{k \in \mathsf{L}(a), a \in S} \mathsf{L}(a)$$

denotes the union of sets of lengths containing $k$. In other words, $\mathcal{V}_k(S)$ is set of all $m \in \mathbb{N}$ for which there exist $u_1, \ldots, u_k, v_1, \ldots, v_m \in \mathcal{A}(S)$ with $u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_m$.

If the set of distances $\Delta(S)$ is finite, then a mild additional assumption guarantees that the sets $\mathcal{V}_k(S)$ are AAPs (almost arithmetical progressions) for all $k \in \mathbb{N}$ ([7, Theorem 4.2]). Easy examples show that even for numerical monoids or for finitely generated Krull monoids unions of sets of lengths need not be arithmetical progressions. However, if $S$ is a Krull monoid, such that every class contains a prime divisor, then all unions of sets of lengths are arithmetical progressions with difference 1 ([6],[9, Theorem 3.1.3]). The question which numerical monoids satisfy an analogue property is still open in general ([1]). The next result offers a sufficient condition for numerical monoids—which is weaker than all the special cases studied so far—which guarantees that unions of sets of lengths $\mathcal{V}_k(S)$ are arithmetical progressions from a certain $k$ on (see [2, Theorem 6.6]).

**Theorem 3.1.** *Let $S$ be a numerical monoid with $\mathcal{A}(S) = \{n_1, \ldots, n_t\}$ where $t \in \mathbb{N}$, $1 < n_1 < \ldots < n_t$, and $d = \gcd(n_2 - n_1, \ldots, n_t - n_{t-1})$. Suppose that the Diophantine equations*

$$(n_2 - n_1)x_2 + \ldots + (n_t - n_1)x_t = dn_1 \quad \text{and} \quad (n_t - n_1)y_1 + \ldots + (n_t - n_{t-1})y_{t-1} = dn_t$$

*have solutions in the non-negative integers. Then there exists a $k^* \in \mathbb{N}$ such that $\mathcal{V}_k(S)$ is an arithmetical progression with difference $d$ for all $k \ge k^*$, and*

$$\lim_{k \to \infty} \frac{|\mathcal{V}_k(S)|}{k} = \frac{1}{d}\Big(\frac{n_t}{n_1} - \frac{n_1}{n_t}\Big)\,.$$

We pose the following open problem.

**Problem 3.2.** Characterize the numerical monoids $S$ for which there exists a $k^* \in \mathbb{N}$ such that the unions of sets of lengths $\mathcal{V}_k(S)$ are arithmetical progressions for all $k \ge k^*$.

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