CIRM

*Courbure discrète : théorie et applications*

Rencontre organisée par :
Laurent Najman and Pascal Romon

18-22 novembre 2013

Facundo Mémoli

**The Gromov-Hausdorff distance: a brief tutorial on some of its quantitative aspects**

Vol. 3, n<sup>o</sup> 1 (2013), p. 89-96.

<http://acirm.cedram.org/item?id=ACIRM_2013__3_1_89_0>

Centre international de rencontres mathématiques
U.M.S. 822 C.N.R.S./S.M.F.
Luminy (Marseille) France

Actes des rencontres du C.I.R.M.
Vol. **3** n° 1 (2013) 89-96

# The Gromov-Hausdorff distance: a brief tutorial on some of its quantitative aspects

Facundo Mémoli

**Abstract**

We recall the construction of the Gromov-Hausdorff distance. We concentrate on quantitative aspects of the definition and on quantitative properties of the distance .

## 1. Introduction

Modeling datasets as metric spaces seems to be natural for some applications and concepts revolving around the Gromov-Hausdorff distance —a notion of distance between compact metric spaces— provide a useful language for expressing properties of data and shape analysis methods.

These notes are based on a talk given during the conference "Discrete Curvature" held in Luminy in November 2013.

*Notation and background concepts.* The book by Burago, Burago, and Ivanov [2] is a valuable source for many concepts in metric geometry. We refer the reader to that book for any concepts not explicitly defined in these notes.

We let $\mathcal{M}$ denote the collection of all compact metric spaces. Recall that for a given metric space $(X, d_X) \in \mathcal{M}$, its *diameter* is defined as $\mathbf{diam}\,(X) := \max_{x,x'\in X} d_X(x,x')$. Similarly, the *radius* of $X$ is defined as $\mathbf{rad}\,(X) := \min_{x\in X}\max_{x'\in X} d_X(x,x')$.

For a fixed metric space $(Z, d_Z)$, we let $d_{\mathcal{H}}^Z$ denote the Hausdorff distance between (closed) subsets of $Z$.

We will often refer to a metric space $(X, d_X)$ by only $X$, but the notation for the underlying metric will be implicitly understood to be $d_X$. Recall, that a map $\varphi : X \to Y$ between metric spaces $(X, d_X)$ and $(Y, d_Y)$ is an *isometric embedding* if $d_Y(\varphi(x), \varphi(x')) = d_X(x, x')$ for all $x, x' \in X$. The map $\varphi$ is an *isometry* if it is a surjective isometric embedding.

## 2. The definition

The goal is to measure distance between two given abstract compact metric spaces. In general, these two spaces may not be readily given as subsets of a common metric space. In this case, the following construction by Gromov [4] applies.

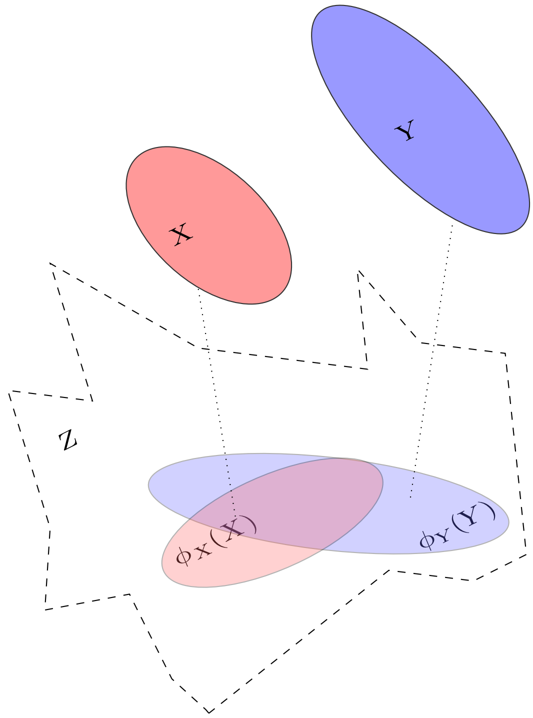
Given $(X, d_X)$ and $(Y, d_Y)$ in $\mathcal{M}$ one considers any "sufficiently rich" third metric space $(Z, d_Z)$ inside which one can find isometric copies of $X$ and $Y$ and measures the Hausdorff distance in $Z$ between these copies. Finally, one minimizes over the choice of the isometric copies and $Z$. Formally, let $Z$, $\phi_X : X \to Z$ and $\phi_Y : Y \to Z$ be respectively a metric space and isometric embeddings of $X$ and $Y$ into $Z$. Then, the Gromov-Hausdorff distance between $X$ and $Y$ is defined as

$$d_{\mathcal{GH}}(X,Y) := \inf_{Z,\phi_X,\phi_Y} d_{\mathcal{H}}^Z\big(\phi_X(X), \phi_Y(Y)\big). \tag{2.1}$$

Text presented during the meeting "Discrete curvature: Theory and applications" organized by Laurent Najman and Pascal Romon. 18-22 novembre 2013, C.I.R.M. (Luminy).

2000 *Mathematics Subject Classification.* 00X99.

*Key words.* metric geometry, graph theory, shape recognition, optimal transportation.




**Theorem 1** ([4]). *$d_{\mathcal{GH}}$ is a legitimate distance on the collection of isometry classes of $\mathcal{M}$.*

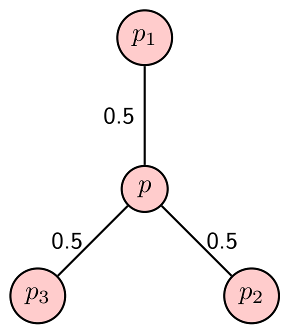
From the practical point of view this definition might not look appealing. As we recall below, there are other more computational suggestive equivalent definitions whose implementation has been explored. But now we try to interpret the definition we have given so far.

2.1. **An example.** Consider the metric spaces $X$ consisting exactly of three points at distance 1 from each other, and $Y$ consisting of exactly one point. Notice that $X$ and $Y$ can be *simultaneously* embedded into $\mathbb{R}^2$ in an isometric way so that $Z = \mathbb{R}^2$ is a valid choice in (2.1) above. The maps $\phi_X$ and $\phi_Y$ represent the relative positions of $X$ and $Y$ in the plane.

By homogeneity, we can assume that the embedding of $X$ is fixed. When choosing $\phi_Y$ one notices that the optimal relative position of $q := \phi_Y(Y)$ with respect to $\Delta := \phi_X(X)$ happens when $q$ is the center of the (equilateral) triangle $\Delta$. In that case, the Hausdorff distance in (2.1) is $\delta_0 := \frac{1}{\sqrt{3}}$ and we conclude that $d_{\mathcal{GH}}(X,Y) \leq \delta_0$. One would be tempted to think that $\delta_0$ is in fact *equal* to Gromov-Hausdorff distance between $X$ and $Y$ but this is not the case!

The same construction that we did above for $\mathbb{R}^2$ can in fact be done on the model hyperbolic two-dimensional space $\mathbb{H}_\kappa$ of curvature $-\kappa$ for *any* $\kappa \leq 0$. As $\kappa \to -\infty$, the (geodesic interpolation of the) triangle $\Delta$ becomes 'thinner' and intuitively, the Hausdorff distance $\delta_\kappa$ between the optimal embeddings in $\mathbb{H}_\kappa$ will decrease as $\kappa$ decreases.

One can in fact consider the following target metric space: $Z_\infty$ consists of four points $p_1, p_2, p_3$, and $p$ such that $d_Z(p_i, p_j) = 1$ for $i \neq j$ and $d_Z(p_i, p) = \frac{1}{2}$ for all $i$. This metric space with four points can be regarded as a subset of the *real tree* (geodesic) metric space below:



This metric space can be regarded as an extreme case of the construction involving the $\mathbb{H}_\kappa$ that was described above. The interesting fact is that if we let $\phi_X(X) = \{p_1, p_2, p_3\}$ and $\phi_Y(Y) = \{q\}$, then $\delta_\infty := d_{\mathcal{H}}^{Z_\infty}\big(\phi_X(X), \phi_Y(Y)\big) = \frac{1}{2}$ which is strictly smaller than $\delta_0$! and thus proves that

$$d_{\mathcal{GH}}(X, Y) \leq \frac{1}{2} < \frac{1}{\sqrt{3}}.$$

One can in fact check that $\delta_\infty < \delta_\kappa \leq \delta_0$ for all $\kappa \in [0, \infty)$. In any case, as we recall in Corollary 5 below, $d_{\mathcal{GH}}(X, Y)$ is always bounded below by $\frac{1}{2}|\mathbf{diam}\,(X) - \mathbf{diam}\,(Y)\,|$. Since in the present case $\mathbf{diam}\,(X) = 1$ and $\mathbf{diam}\,(Y) = 0$, we obtain that $d_{\mathcal{GH}}(X, Y) \geq \frac{1}{2}$ which together with the reverse inequality obtained above implies that in fact, for the example under consideration, $d_{\mathcal{GH}}(X, Y) = \frac{1}{2}$!

2.2. **A simplification.** Kalton and Ostrovskii [5] observed that one can *equivalently* define the Gromov-Hausdorff distance between $X$ and $Y$ by considering $Z$ in (2.1) to be the disjoint union $X \sqcup Y$ together with *any* metric d such that $\mathrm{d}_{|X\times X} = d_X$ and $\mathrm{d}_{|Y\times Y} = d_Y$. Let $\mathcal{D}(d_X, d_Y)$ denote the set of *all* such metrics on $X \sqcup Y$. Then, they observe that

$$d_{\mathcal{GH}}(X, Y) = \inf_{\mathrm{d}\in\mathcal{D}(d_X, d_Y)} d_{\mathcal{H}}^{(X\sqcup Y, \mathrm{d})}\big(X, Y\big). \tag{2.2}$$

This expression for the Gromov-Hausdorff distance seems more appealing for the computationally minded: imagine that $X$ and $Y$ are finite, then the *variable* d in the underlying optimization problem can be regarded as a matrix in $\mathbb{R}^{|X|\times|Y|}$. If we assume that $|X| = |Y| = n$ then the number of *linear constraints* that each d in $\mathcal{D}(d_X, d_Y)$ must satisfy is of order $n^3$ (all triangle inequalities). Even more explicitly, the optimization problem over $\mathcal{D}(d_X, d_Y)$ that one must solve in practice is (cf. [7]) $\min_{\mathrm{d}} J(\mathrm{d})$ where

$$J(\mathrm{d}) := \max\left(\max_{x\in X}\min_{y\in Y}\mathrm{d}(x, y), \max_{y\in Y}\min_{x\in X}\mathrm{d}(x, y)\right).$$

The complexity from the original definition (2.1) is now hidden in the fact that $J(\cdot)$ is highly non-linear.

Going back to the example discussed in 2.1, one can state that in the context of (2.2), the optimal metric on $X \sqcup Y$ is

$$\mathrm{d}^* := \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

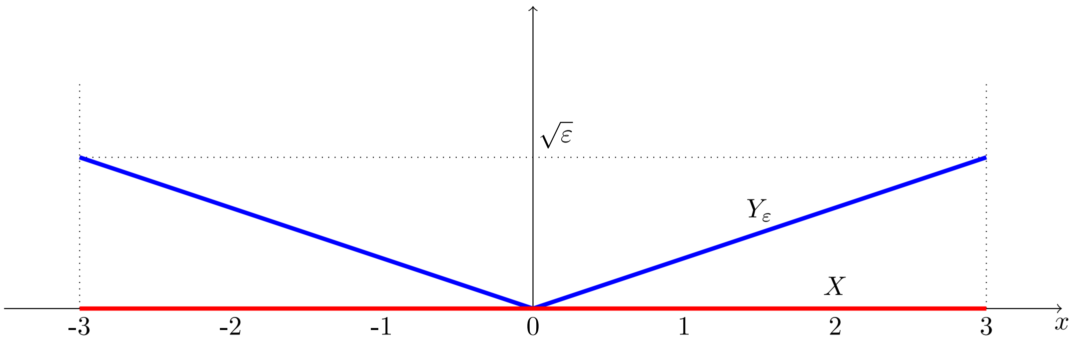
2.3. **The case of subsets of Euclidean space.** Even if we saw in Section 2.1 above that when $X$ and $Y$ are subsets of $\mathbb{R}^d$ the optimal $Z$ in (2.1) may not be $\mathbb{R}^d$, one can still relate $d_{\mathcal{GH}}(X, Y)$ with some natural notion of distance for subsets of Euclidean space. Doing this provides more insight into as to how the Gromov-Hausdorff distance operates in situations for which we already have a well developed intuition.

An *intrinsic approach* to comparing two subsets $X$ and $Y$ of $\mathbb{R}^d$ would be to regard them as metric spaces by endowing them with the restriction of the ambient space metric: $d_X(\cdot, \cdot) = \|\cdot - \cdot\|$ etc. So, one can consider $d_{\mathcal{GH}}(X, Y)$ as a possible notion of dissimilarity between $X$ and $Y$.

Another notion of dissimilarity that is frequently considered in shape and data analysis arises from the Hausdorff distance modulo rigid isometries and constitutes an *extrinsic approach*: let $E(d)$ denote the group of isometries of $\mathbb{R}^d$ and define

$$d_{\mathcal{H}}^{\mathbb{R}^d,\mathrm{rigid}}(X, Y) := \inf_{T\in E(d)} d_{\mathcal{H}}^{\mathbb{R}^d}\big(X, T(Y)\big).$$

Since in this case, one can always choose $Z = \mathbb{R}^d$ in (2.1) above, one immediately sees that $d_{\mathcal{GH}}(X, Y) \leq d_{\mathcal{H}}^{\mathbb{R}^d,\mathrm{rigid}}(X, Y)$. Even if we already saw in Section 2.1 that the equality cannot take place in general, one could hope that for some suitable $C > 0$, $d_{\mathcal{H}}^{\mathbb{R}^d,\mathrm{rigid}}(X, Y) \leq C \cdot d_{\mathcal{GH}}(X, Y)$ for all $X, Y \subset \mathbb{R}^d$ compact. Interestingly, however, this cannot happen! Consider $X = [-1, 1]$. Fix $0 < \varepsilon \ll 1$ and let $f_\varepsilon(x) := |x| \cdot \sqrt{\varepsilon}$. Let $Y_\varepsilon$ be the set $\{(x, f_\varepsilon(x)); x \in [-1, 1]\}$. Notice that $\mathbf{rad}\,(X) = 1$ and $\mathbf{rad}\,(Y_\varepsilon) = \sqrt{1+\varepsilon}$.




In any case, it is clear that for $\varepsilon > 0$ small enough, $d_{\mathcal{H}}^{\mathbb{R}^d,\text{rigid}}(X, Y_\varepsilon) = \frac{\sqrt{\varepsilon}}{2}$. However, since by Proposition 6 and Corollary 4 below,

- $d_{\mathcal{GH}}(X, Y_\varepsilon) \geq \frac{1}{2}|\mathbf{rad}\,(X) - \mathbf{rad}\,(Y_\varepsilon)\,| = \frac{1}{2}(\sqrt{1+\varepsilon} - 1) \geq \frac{\varepsilon}{2+2\sqrt{2}}$ and
- $d_{\mathcal{GH}}(X, Y_\varepsilon) \leq \frac{1}{2}\sup_{|x|\neq|x'|}|x-x'|\cdot\left(\sqrt{1+\varepsilon\cdot\left(\frac{|x|-|x'|}{x-x'}\right)^2}-1\right) \leq \varepsilon$, since $\big||x|-|x'|\big| \leq |x-x'|$ for all $x, x' \in X$.

It follows that $d_{\mathcal{GH}}(X, Y_\varepsilon)$ is of order $\varepsilon$ and therefore no constant $C > 0$ will guarantee that $C \cdot d_{\mathcal{GH}}(X, Y_\varepsilon) \geq d_{\mathcal{H}}^{\mathbb{R}^d,\text{rigid}}(X, Y_\varepsilon)$ for all $1 \gg \varepsilon > 0$!

What does hold for this construction is that $C \cdot \big(d_{\mathcal{GH}}(X, Y_\varepsilon)\big)^{1/2} \geq d_{\mathcal{H}}^{\mathbb{R}^d,\text{rigid}}(X, Y_\varepsilon)$ for some constant $C > 0$. It turns out that this is not an isolated phenomenon:

**Theorem 2** ([6])**.** *For each natural number $d \geq 2$ there exists $c_d > 0$ such that for all $X, Y \in \mathbb{R}^d$ one has*

$$d_{\mathcal{GH}}(X, Y) \leq d_{\mathcal{H}}^{\mathbb{R}^d,\text{rigid}}(X, Y) \leq c_d \cdot M^{1/2} \cdot \big(d_{\mathcal{GH}}(X, Y)\big)^{1/2},$$

*where* $M = \max(\mathbf{diam}\,(X), \mathbf{diam}\,(Y))$.

2.4. **Another expression and consequences.** For two sets $X$ and $Y$ let $\mathcal{R}(X, Y)$ denote the set of all *correspondences* between $X$ and $Y$, that is, sets $R \subseteq X \times Y$ such that $\pi_1(R) = X$ and $\pi_2(R) = Y$. In general, we will refer to any non-empty set $R$ of $X \times Y$ as a *relation* between $X$ and $Y$. Obviously, all correspondences are relations.

The *distortion* of a relation $R$ between the metric spaces $(X, d_X)$ and $(Y, d_Y)$ is defined as the number

$$\text{dis}(R) := \sup_{(x,y),(x',y')\in R} \big|d_X(x, x') - d_Y(y, y')\big|.$$

Notice that given a function $\varphi : X \to Y$ one can define the relation $R_\varphi := \{(x, \varphi(x)); x \in X\}$, and in that case we write $\text{dis}(\varphi) := \text{dis}(R_\varphi) = \sup_{x,x'\in X}\big|d_X(x, x') - d_Y(\varphi(x), \varphi(x'))\big|$. Similarly, when $\psi : Y \to X$ is given, it induces the relation $R_\psi := \{(\psi(y), y); y \in Y\}$. Note that the structure of $R_\varphi$ is different from the structure of $R_\psi$.

Now, when a map $\varphi : X \to Y$ and a map $\psi : Y \to X$ are both specified, we consider the relation $R_{\varphi,\psi} := R_\varphi \bigcup R_\psi$ and note that in fact $R_{\varphi,\psi}$ is actually a correspondence between $X$ and $Y$.

Furthermore, one can explicitly compute that

$$\text{dis}(R_{\varphi,\psi}) = \max\big(\text{dis}(\varphi), \text{dis}(\psi), C(\varphi, \psi)\big),$$

where $C(\varphi, \psi) := \sup_{x\in X, y\in Y}\big|d_X(x, \psi(y)) - d_Y(\varphi(x), y)\big|$. Notice that if $C(\varphi, \psi) < \eta$ for some $\eta > 0$, then $\big|d_X(x, \psi(y)) - d_Y(\varphi(x), y)\big| < \eta$ for all $(x, y) \in X \times Y$. In particular, for $x = \psi(y)$, it follows that $d_Y(\varphi \circ \psi(y), y) < \eta$ for all $y \in Y$. Similarly one can obtain $d_X(x, \psi \circ \varphi(x)) < \eta$ for all $x \in X$. These two conditions are often interpreted as meaning that $\varphi$ and $\psi$ are close to being inverses of each other. This proximity is quantified by $\eta$.

An interesting and useful characterization of the Gromov-Hausdorff distance based on optimization over correspondences is the following:

**Theorem 3** ([5])**.** *For all $X, Y \in \mathcal{M}$ one has that*

$$d_{\mathcal{GH}}(X, Y) \overset{\text{(I)}}{=} \frac{1}{2}\inf_{R\in\mathcal{R}(X,Y)} \text{dis}(R) \overset{\text{(II)}}{=} \frac{1}{2}\inf_{\varphi,\psi} \text{dis}(R_{\varphi,\psi}).$$

**Corollary 4.** *Let $X$ be a set and $d$ and $d'$ be any two metrics on $X$. Then,*

$$d_{\mathcal{GH}}((X,d),(X,d')) \leq \frac{1}{2} \sup_{x,x' \in X} \big|d(x,x') - d'(x,x')\big|.$$

The theorem above is significant for several reasons. First of all, (I) indicates that solving for the Gromov-Hausdorff distance between two finite metric spaces is an instance of a well known combinatorial optimization problem called the *bottleneck quadratic assignment problem* or bQAP. The bQAP is NP-Hard and furthermore, computing any $(1+\varepsilon)$ of the optimal solution is also NP-Hard for any $\varepsilon > 0$ [12]. See [9, 10, 1] for some heuristic approaches.

A second observation stemming from the equality (II) in the theorem is the fact that since the term $C(\varphi,\psi)$ acts as a *coupling term* in the optimization

$$d_{\mathcal{GH}}(X,Y) = \frac{1}{2} \inf_{\varphi,\psi} \max\big(\mathrm{dis}(\varphi), \mathrm{dis}(\psi), C(\varphi,\psi)\big),$$

one could conceive of dropping it from the expression above yielding

$$d_{\mathcal{GH}}(X,Y) \geq \frac{1}{2} \max\big(\inf_{\varphi} \mathrm{dis}(\varphi), \inf_{\psi} \mathrm{dis}(\psi)\big) =: \widehat{d_{\mathcal{GH}}}\,(X,Y)\,.$$

It is important to notice that computing $\widehat{d_{\mathcal{GH}}}\,(X,Y)$, which we call the *modified Gromov-Hausdorff distance* [8], leads to solving two *decoupled* optimization problems, a feature which is desirable in applications. However, the computational complexity of the problems of the type $\inf_\varphi \mathrm{dis}(\varphi)$ could still be high. We will explore some interesting structure that arises from this modified definition in the next section but for now we will make one more observation based on the expression given by Theorem 3.

From equality (I) it follows that the Gromov-Hausdorff distance between any compact metric space and the metric space consisting of exactly one point is $d_{\mathcal{GH}}(X,*) = \frac{1}{2}\mathbf{diam}\,(X)$. As a corollary from Theorem 1 and this observation one has

**Corollary 5.** *For all $X,Y \in \mathcal{M}$, $d_{\mathcal{GH}}(X,Y) \geq \frac{1}{2}\big|\mathbf{diam}\,(X) - \mathbf{diam}\,(Y)\,\big|$.*

*Proof.* The inequality $d_{\mathcal{GH}}(X,Y) \geq \big|d_{\mathcal{GH}}(X,*) - d_{\mathcal{GH}}(Y,*)\big|$ is guaranteed by the triangle inequality for the Gromov-Hausdorff distance. The remark preceding the statement completes the proof. $\square$

A similar lower bound for the Gromov-Hausdorff distance arises from considering the radius of metric spaces:

**Proposition 6** ([8])**.** *For all $X,Y \in \mathcal{M}$, $d_{\mathcal{GH}}(X,Y) \geq \frac{1}{2}\big|\mathbf{rad}\,(X) - \mathbf{rad}\,(Y)\,\big|$.*

## 3. The modified Gromov-Hausdorff and curvature sets

It could appear plausible that by dropping the coupling term $C(\varphi,\psi)$ in the optimization above one might have lost some of the nice theoretical properties enjoyed by the Gromov-Hausdorff distance. This is not the case, and in fact the modified Gromov-Hausdorff retains many of these good properties:

**Theorem 7** ([8])**.** *The modified Gromov-Hausdorff distance satisfies:*

1. *$\widehat{d_{\mathcal{GH}}}$ is a legitimate metric on the isometry classes of $\mathcal{M}$.*
2. *$d_{\mathcal{GH}}(X,Y) \geq \widehat{d_{\mathcal{GH}}}\,(X,Y)$ for all $X,Y \in \mathcal{M}$.*
3. *$d_{\mathcal{GH}}$ and $\widehat{d_{\mathcal{GH}}}$ are topologically equivalent within $d_{\mathcal{GH}}$-precompact families of $\mathcal{M}$.*

It is however interesting that the equality in item (2) *does not take place in general.* In fact, [8] provides a counterexample.

### 3.1. Curvature sets.

Gromov [4] defines for each $n \in \mathbb{N}$ the curvature sets of $X \in \mathcal{M}$ in the following way: let $\Psi_X^{(n)} : X^{\times n} \to \mathbb{R}^{n\times n}$ be the matrix valued map defined by $(x_1,\ldots,x_n) \mapsto ((d_X(x_i,x_j)))_{i,j=1}^n$. This map simply assigns to each $n$-tuple of points its *distance matrix*: the matrix arising from restricting the metric on $X$ to the given $n$-tuple. Then, the $n$-th curvature set of $X$ is

$$\mathbf{K}_n(X) := \big\{\Psi_X^{(n)}(x_1,\ldots,x_n); (x_1,\ldots,x_n) \in X^{\times n}\big\}.$$

In colloquial terms, curvature sets are just 'bags' containing all the possible distance matrices of a given size arising from points sampled from $X$.

For example, when $n = 2$, $\mathbf{K}_2(X)$ contains the same information as $\{d_X(x, x'); x, x' \in X\} \subset \mathbb{R}_+$. In contrast, $\mathbf{K}_3(X)$ contains all 'triangles' from $X$ and this particular case suggest one possible justification for the name 'curvature sets'. Indeed, let $X$ be a smooth planar curve. Consider any three points $x_1, x_2$ and $x_3$ on $X$ close to each other. Then, if $a = \|x_2 - x_1\|$, $b = \|x_1 - x_3\|$, and $c = \|x_1 - x_2\|$, the inverse of the radius $R$ of the circle circumscribed to the triangle $\Delta x_1 x_2 x_3$ admits an explicit expression in terms of $a, b$ and $c$: $R^{-1} = \frac{4\,S(a,b,c)}{a\,b\,c}$ where $S(a, b, c)$ is the area of the triangle as given by Heron's formula.[1] The crucial observation is that $R$ can be computed exclusively from the information contained in $\mathbf{K}_3(X)$. Now, by an argument involving a series expansion [3], as $a, b, c \to 0$ $R^{-1}$ converges to the curvature $\kappa$ of $X$ at the point of coalescence of $x_1, x_2, x_3$.

Curvature sets absorb all the information that one needs in order to determine whether two compact metric spaces are isometric or not.

**Theorem 8** ([4]). *Let $X, Y \in \mathcal{M}$. Then, $X$ and $Y$ are isometric if and only if $\mathbf{K}_n(X) = \mathbf{K}_n(Y)$ for all $n \in \mathbb{N}$.*

Constructions similar to curvature sets have also been considered by Peter Olver in the context of subsets of Euclidean space [11].

*An example: Curvature sets of spheres.* We illustrate the definition with an example from [8]. Consider first the case of the standard circle $S^1$ endowed with the angular distance. We will exactly characterize $\mathbf{K}_3(S^1)$. For that purpose first consider any embedding of $S^1$ into $\mathbb{R}^2$ and observe that for any three points on $S^1$ exactly one the following two conditions holds: (a) there exists a line through the center of the circle such that the three points are contained on one side of the line; (b) no such line exists.

Case (a) means that one of the three distances defined by the three points must forcibly be equal to the sum of the other two distances. Case (b) implies that the sum of the three distances is exactly $2\pi$. Also note that, by symmetry, case (a) unrolls into three different cases depending on the identity of the distance that is equal to the sum of the other two. Each of these four situations gives a *linear* relation between the three distances! Thus, we obtain that $\mathbf{K}_3(S^1)$ is isomorphic to the tetrahedron with vertices $(0, 0, 0)$, $(0, \pi, \pi)$, $(\pi, 0, \pi)$, and $(\pi, \pi, 0)$.

The case of $S^2$, when endowed with the standard geodesic distance, is similar and one can prove that $\mathbf{K}_3(S^2)$ is the *convex hull* of $\mathbf{K}_3(S^1)$.

3.2. **Comparing curvature sets?** An interesting property of curvature sets is that they are *isometry invariants* of metric spaces which 'live' in fixed target spaces. More precisely, for any $X, Y \in \mathcal{M}$, $\mathbf{K}_n(X)$ and $\mathbf{K}_n(Y)$ are both subsets of $\mathbb{R}^{n\times n}$.

With the purpose of discriminating $X$ and $Y$ one may conceive of comparing $\mathbf{K}_n(X)$ and $\mathbf{K}_n(Y)$. Since they are both (compact) sub-sets of $\mathbb{R}^{n\times n}$ one could compute the Hausdorff distance between them. For this we first endow $\mathbb{R}^{n\times n}$ with the distance $d_{\ell^\infty}(A, B) := \max_{i,j} |a_{i,j} - b_{i,j}|$ for $A = (\!(a_{i,j})\!)$ and $B = (\!(b_{i,j})\!)$ in $\mathbb{R}^{n\times n}$. Then, we compute

$$d_n(X, Y) := \frac{1}{2}\, d_{\mathcal{H}}^{\mathbb{R}^{n\times n}}\big(\mathbf{K}_n(X), \mathbf{K}_n(Y)\big),$$

and use this number as an indication of how similar $X$ and $Y$ are. The best possible measure of dissimilarity that this sort of idea suggests is to consider

$$d_\infty(X, Y) := \sup_{n\in\mathbb{N}} d_n(X, Y).$$

Theorem 8 guarantees that $d_\infty$ defines a legitimate metric on $\mathcal{M}$ modulo isometries.

Interestingly, one has the following 'structural theorem' for the modified Gromov-Hausdorff distance in terms of curvature sets:

**Theorem 9** ([8]). *For all $X, Y \in \mathcal{M}$, $\widehat{d}_{\mathcal{GH}}(X, Y) = d_\infty(X, Y)$.*

---

[1] $S(a, b, c) = \frac{1}{4}\big((a + b + c)(a - b + c)(a + b - c)(-a + b + c)\big)^{1/2}$.

This theorem provides a useful path for computing estimates to the Gromov-Hausdorff distance. Furthermore, the theorem suggests a way of 'slicing' the computation/approximation of the Gromov-Hausdorff distance between finite metric spaces, since one might want to consider computing $d_n$ for a fixed $n$ and hope that this provides enough information for discriminating spaces within a given family. For finite spaces, the computation of $d_n$ would incur a polynomial cost, albeit of a high order. There are some known classes of metric spaces $\mathcal{C} \subset \mathcal{M}$ that are characterized up to isometry by $\mathbf{K}_n(\cdot)$ for some finite $n = n(\mathcal{C})$, see [8].

*A lower bound for $d_{\mathcal{GH}}(S^1, S^2)$.* Theorems 7 item (2) and 9 then guarantee that

$$d_{\mathcal{GH}}(S^1, S^2) \geq d_3(S^1, S^2) = \frac{1}{2} d_{\mathcal{H}}^{\mathbb{R}^{3\times 3}}\big(\mathbf{K}_3(S^1), \mathbf{K}_3(S^2)\big) =: \xi.$$

Since $\mathbf{K}_3(S^2)$ is the convex hull of $\mathbf{K}_3(S^1)$, $\mathbf{K}_3(S^1) \subset \mathbf{K}_3(S^2)$, and therefore,

$$\xi = \frac{1}{2} \max_{p \in \mathbf{K}_3(S^2)} \min_{q \in \mathbf{K}_3(S^1)} \|p - q\|_\infty = \min_{q \in \mathbf{K}_3(S^1)} \|g - q\|,$$

where $g = \frac{\pi}{2}(1,1,1)$ is the center of $\mathbf{K}_3(S^2)$. But now, the center $c = \frac{2\pi}{3}(1,1,1)$ of the face of $\mathbf{K}_3(S^1)$ determined by $\pi\,(0,1,1)$, $\pi\,(1,0,1)$, and $\pi\,(1,1,0)$ is at minimal $\ell^\infty$ distance from $g$ so that $\xi = \frac{1}{2}|\frac{\pi}{2} - \frac{2\pi}{3}| = \frac{\pi}{12}$, and we find the lower bound $d_{\mathcal{GH}}(S^1, S^2) \geq \frac{\pi}{12}$.

## 4. Discussion and outlook

The Gromov-Hausdorff distance offers a useful language for expressing different tasks in shape and data analysis. Its origins are in the work of Gromov on synthetic geometry. For finite metric spaces, the Gromov-Hausdorff distance leads to solving NP-Hard combinatorial optimization problems. A related to construction is that of *Gromov-Wasserstein* distances which operate on metric measure spaces [13, 7]. In contrast to the Gromov-Hausdorff distance, the computation of Gromov-Wasserstein distances leads to solving quadratic optimization problems on *continuous* variables. The space of all metric measures spaces endowed with a certain variant of the Gromov-Wasserstein distance [7] enjoys nice theoretical properties [14]. It seems of interest to develop provably correct approximations to these distances when restricted to some suitable subclasses of finite metric spaces.

## References


[1] Alexander M. Bronstein, Michael M. Bronstein, and Ron Kimmel. Efficient computation of isometry-invariant distances between surfaces. *SIAM Journal on Scientific Computing*, 28(5):1812–1836, 2006.

[2] D. Burago, Y. Burago, and S. Ivanov. *A Course in Metric Geometry*, volume 33 of *AMS Graduate Studies in Math.* American Mathematical Society, 2001.

[3] Eugenio Calabi, Peter J. Olver, Chehrzad Shakiban, Allen Tannenbaum, and Steven Haker. Differential and numerically invariant signature curves applied to object recognition. *Int. J. Comput. Vision*, 26(2):107–135, 1998.

[4] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, 1999.

[5] N. J. Kalton and M. I. Ostrovskii. Distances between Banach spaces. *Forum Math.*, 11:1:17–48, 1999.

[6] Facundo Mémoli. Gromov-Hausdorff distances in Euclidean spaces. *Computer Vision and Pattern Recognition Workshops, 2008. CVPR Workshops 2008. IEEE Computer Society Conference on*, pages 1–8, June 2008.

[7] Facundo Mémoli. Gromov-Wasserstein distances and the metric approach to object matching. *Foundations of computational mathematics*, 11:417–487, 2011.

[8] Facundo Mémoli. Some properties of Gromov—Hausdorff distances. *Discrete & Computational Geometry*, pages 1–25, 2012. 10.1007/s00454-012-9406-8.

[9] Facundo Mémoli and Guillermo Sapiro. Comparing point clouds. In *SGP '04: Proceedings of the 2004 Eurographics/ACM SIGGRAPH symposium on Geometry processing*, pages 32–40, New York, NY, USA, 2004. ACM.

[10] Facundo Mémoli and Guillermo Sapiro. A theoretical and computational framework for isometry invariant recognition of point cloud data. *Found. Comput. Math.*, 5(3):313–347, 2005.

[11] P.J. Olver. Joint invariant signatures. *Foundations of computational mathematics*, 1(1):3–68, 2001.

[12] Panos M. Pardalos and Henry Wolkowicz, editors. *Quadratic assignment and related problems.* DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 16. American Mathematical Society, Providence, RI, 1994. Papers from the workshop held at Rutgers University, New Brunswick, New Jersey, May 20–21, 1993.

[13] Karl-Theodor Sturm. On the geometry of metric measure spaces. I. *Acta Math.*, 196(1):65–131, 2006.
[14] Karl-Theodor Sturm. The space of spaces: curvature bounds and gradient flows on the space of metric measure spaces. *arXiv preprint arXiv:1208.0434*, 2012.

Department of Mathematics, The Ohio State University, Columbus, OH 43210, United States of America • memoli@math.osu.edu