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Discrete complex analysis – the medial graph approach

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Abstract

We discuss a new formulation of the linear theory of discrete complex analysis on planar quad-graphs based on their medial graphs. It generalizes the theory on rhombic quad-graphs developed by Duffin, Mercat, Kenyon, Chelkak and Smirnov and follows the approach on general quad-graphs proposed by Mercat. We provide discrete counterparts of the most fundamental objects in complex analysis such as holomorphic functions, differential forms, derivatives, and the Laplacian. Also, we discuss discrete versions of important fundamental theorems such as Green's identities and Cauchy's integral formulae. For the first time, Green's first identity and Cauchy's integral formula for the derivative of a holomorphic function are discretized.

1. History

Discrete harmonic functions on the square lattice were studied by a number of authors in the 1920s, including Courant, Friedrichs, and Lewy [5]. Discrete holomorphic functions on the square lattice were studied by Isaacs [10]. He proposed two different definitions for holomorphicity. One of them was reintroduced and further investigated by Lelong-Ferrand [8]. She developed the theory to a level that allowed her to prove the Riemann mapping theorem using discrete methods [13]. Duffin also studied discrete complex analysis on the square grid [6], and he was the first who extended the theory to rhombic lattices [7]. Kenyon [12], and Chelkak and Smirnov [3] resumed the investigation of discrete complex analysis on rhombic lattices, or, equivalently, isoradial graphs.

Mercat extended the theory from domains in the complex plane to discrete Riemann surfaces, first considering cellular decompostions into rhombi [14] and later generalizing the notions to general quadrilaterals [16]. The motivation for this theory of discrete Riemann surfaces is derived from statistical physics, in particular, the Ising model. Mercat defined a discrete Dirac operator and discrete spin structures, and he identifies criticality in the Ising model with rhombic quad-graphs.

Some two-dimensional discrete models in statistical physics exhibit conformally invariant properties in the thermodynamical limit. Such conformally invariant properties were established by Chelkak and Smirnov for the Ising model [4], and by Kenyon for the dimer model on a square grid [11]. In both cases, linear theories of discrete analytic functions on regular grids were highly important. Kenyon, Chelkak and Smirnov obtained important analytic results [12, 3], which were instrumental in the proof that the critical Ising model is universal [4].

Important non-linear discrete theories of complex analysis involve circle packings, or, more generally, circle patterns. Rodin and Sullivan first proved that the Riemann mapping of a complex domain to the unit disk can be approximated by circle packings [17]. A similar result for isoradial

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circle patterns, even with irregular combinatorics, is due to Bücking [2]. The first author, Mercat and Suris showed how the linear theory of discrete holomorphic functions on quad-graphs can be obtained by linearizing the theory on circle patterns: Discrete holomorphic functions describe infinitesimal deformations of circle patterns [1].

2. Organization of the paper

Our setup is a strongly regular cellular decompostion of the complex plane into quadrilaterals, called quad-graph, which we assume to be bipartite. Basic notations for quad-graphs used in this paper are introduced in Section 3. Of crucial importance for our work is the medial graph of a quad-graph in Section 4. It provides the connection between the notions of discrete derivatives of Kenyon [12], Mercat [15], and Chelkak and Smirnov [3], extended from rhombic to general quad-graphs, and discrete differential forms and discrete exterior calculus. We discuss the discrete derivatives in Sections 5 and 7. Concerning discrete differential forms in Section 6, we get essentially the same definitions as Mercat proposed in [16]. However, our notation of discrete exterior calculus in Sections 8, 9, and 10 is slightly more general and shows its power when considering integral formulae. In Section 11, we discuss the discrete Laplacian introduced by Mercat [16]. In particular, we prove discrete Green's identities and recover the factorization of the discrete Laplacian known from the rhombic case [12, 15]. We formulate discrete Cauchy's integral formulae for discrete holomorphic functions and their discrete derivatives in Section 12.

To keep the paper short, we highlight just the most instructive proofs, and omit the others. However, the skipped proofs are usually elementary calculations or immediate consequences of previous statements. The proofs and a more detailed discussion of discrete complex analysis on planar quad-graphs can be found in the dissertation of the second author [9]. There, we also investigate discrete Green's functions, prove their existence and the existence of discrete Cauchy's kernels, and provide several results concerning the asymptotics of these functions in the case of certain parallelogram-graphs.

3. Bipartite quad-graphs

We consider a strongly regular and locally finite cellular decompostion of the complex plane $\mathbb{C}$ into quadrilaterals, described by a bipartite *quad-graph* Λ. The sets of vertices, edges, and faces, are denoted by $V(\Lambda)$, $E(\Lambda)$, and $F(\Lambda)$, respectively. We refer to the maximal independent sets of vertices of Λ as *black* and *white* vertices. Let Γ and Γ^* be the graphs defined on the black and white vertices where the edges are exactly the diagonals of faces of Λ. It is easy to see that Γ and Γ^* are dual to each other. For the ease of notation, we identify the vertices of Λ with their corresponding complex values, and to oriented edges of $\Lambda, \Gamma, \Gamma^*$ we assign the complex numbers determined by the difference of their two endpoints.

To Λ we associate its *dual* $\Diamond = \Lambda^*$. In this paper, we look at $\Diamond$ in an abstract way, identifying vertices or faces of $\Diamond$ with corresponding faces or vertices of Λ, respectively. However, in the particular case that all quadrilaterals are parallelograms, it makes sense to place the vertices of $\Diamond$ at the centers of the parallelograms [9]. If a vertex $v \in \Lambda$ is a vertex of a quadrilateral $Q \in \Diamond$, we write $Q \sim v$ or $v \sim Q$ and call v and Q *incident* to each other. The vertices of Q are denoted by b_-, w_-, b_+, w_+ in counterclockwise order, where $b_\pm \in \Gamma$ and $w_\pm \in \Gamma^*$.

Definition 1. For a quadrilateral $Q \in V(\Diamond) \cong F(\Lambda)$ we define

$$\rho(b_-, b_+) = \rho(b_+, b_-) := -i\frac{w_+ - w_-}{b_+ - b_-} =: \frac{1}{\rho(w_-, w_+)} = \frac{1}{\rho(w_+, w_-)}.$$

Let $\varphi_Q := \arccos\left(\mathrm{Re}\left(i\rho(b_-, b_+)\right)\right)$ be the angle under which the diagonal lines of Q intersect.

Figure 3.1 shows a finite bipartite quad-graph together with the notation for a single quadrilateral Q and the *star of a vertex* v, i.e., the set of all faces incident to v.

In addition, we denote by $\Diamond_0$ a connected subset of $\Diamond$. It is called *simply-connected* if the corresponding set of cells in $\mathbb{C}$ is simply-connected. Its vertices induce subgraphs Λ_0 of Λ, Γ_0 of Γ, and Γ_0^* of Γ^*. For simplicity, we always assume that the induced subgraphs are connected as well.

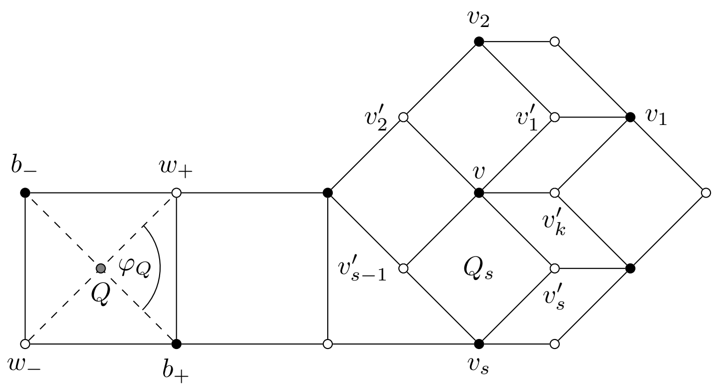

Figure 3.1: Bipartite quad-graph with notations

4. Medial graph

Definition 2. The *medial graph* X of Λ is defined as follows. Its vertex set is given by all the midpoints of edges of Λ, and two vertices are adjacent iff the corresponding edges belong to the same face and have a vertex in common. The set of faces of X is in bijective correspondence with $V(\Lambda) \cup V(\Diamond)$: The vertices of a face F_v corresponding to $v \in V(\Lambda)$ are the midpoints of edges of Λ incident to v, and the vertices of a quadrilateral face F_Q corresponding to $Q \in V(\Diamond)$ are the midpoints of the four edges of Λ belonging to Q.

Any edge e of X is the common edge of two faces F_Q and F_v for $Q \sim v$, denoted by $[Q, v]$.

Let $Q \in V(\Diamond)$ and $v_0 \sim Q$. Due to Varignon's theorem, F_Q is a parallelogram, and the complex number assigned to the edge $e = [Q, v_0]$ connecting the midpoints of edges $v_0v'_-$ and $v_0v'_+$ of Λ is just half of $e = v'_+ - v'_-$. In Figure 4.1, showing Λ with its medial graph, the vertices of F_Q and F_v, $v \in V(\Lambda)$, are colored gray.

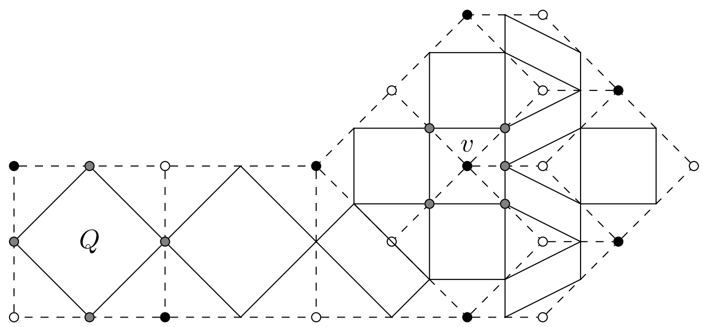

Figure 4.1: Bipartite quad-graph (dashed) with medial graph (solid)

For a subgraph $\Diamond_0 \subseteq \Diamond$, we denote by $X_0 \subseteq X$ the subgraph of X whose edges are contained in faces of $\Diamond_0$. Note that the medial graph X corresponds to a (strongly regular and locally finite) cellular decompostion of $\mathbb{C}$ in a canonical way. In particular, we can talk about a topological disk in $F(X_0)$ and about a (counterclockwise oriented) boundary ∂X_0.

Definition 3. For $v \in V(\Lambda)$ and $Q \in V(\Diamond)$, let P_v and P_Q be the closed paths on X connecting the midpoints of edges of Λ incident to v and Q, respectively, in counterclockwise direction. In Figure 4.1, their vertices are colored gray. We call P_v and P_Q *discrete elementary cycles.*

5. Discrete derivatives of functions on the vertices of the quad-graph

In the classical theory, holomorphic functions (with nowhere-vanishing derivative) preserve angles, and at a single point, lengths are uniformly scaled. This motivates the following definition of discrete holomorphicity [16] that was also used previously in the rhombic setting.

Definition 4. Let $Q \in V(\Diamond)$ and f a complex function on b_-, w_-, b_+, w_+. f is called *discrete holomorphic* at Q if it satisfies the *discrete Cauchy-Riemann equation*

$$\frac{f(b_+) - f(b_-)}{b_+ - b_-} = \frac{f(w_+) - f(w_-)}{w_+ - w_-}.$$

For discrete holomorphicity, only the differences on Γ and Γ^* matter. Hence, we should not consider constants on $V(\Lambda)$, but biconstants [15] determined by each a value on $V(\Gamma)$ and $V(\Gamma^*)$. We call functions that are constant on $V(\Gamma)$ and constant on $V(\Gamma^*)$ *biconstant*.

Definition 5. Let $Q \in V(\Diamond)$, and let f be a complex function on b_-, w_-, b_+, w_+. The *discrete derivatives* $\partial_\Lambda f$, $\bar{\partial}_\Lambda f$ are defined by

$$\partial_\Lambda f(Q) := \frac{\exp\left(-i\left(\varphi_Q - \frac{\pi}{2}\right)\right)}{2\sin(\varphi_Q)} \cdot \frac{f(b_+) - f(b_-)}{b_+ - b_-} + \frac{\exp\left(i\left(\varphi_Q - \frac{\pi}{2}\right)\right)}{2\sin(\varphi_Q)} \cdot \frac{f(w_+) - f(w_-)}{w_+ - w_-},$$

$$\bar{\partial}_\Lambda f(Q) := \frac{\exp\left(i\left(\varphi_Q - \frac{\pi}{2}\right)\right)}{2\sin(\varphi_Q)} \cdot \frac{f(b_+) - f(b_-)}{\overline{b_+ - b_-}} + \frac{\exp\left(-i\left(\varphi_Q - \frac{\pi}{2}\right)\right)}{2\sin(\varphi_Q)} \cdot \frac{f(w_+) - f(w_-)}{\overline{w_+ - w_-}}.$$

In the case of quadrilaterals whose diagonals intersect orthogonally, $\varphi_Q = \pi/2$, and $\partial_\Lambda f, \bar{\partial}_\Lambda f$ are exactly defined as in [3]. They naturally discretize their smooth counterparts $(\partial_x - i\partial_y)/2$ and $(\partial_x + i\partial_y)/2$. In a general quadrilateral Q, we have to take the deviation $(\varphi_z - \pi/2)$ from orthogonality into account, and change the factors appropriately.

Proposition 6. *Let $\Diamond_0 \subseteq \Diamond$ and f be a discrete holomorphic function on $V(\Lambda_0)$.*

(1) *If f is purely imaginary or purely real, f is biconstant.*
(2) *If $\bar{\partial}_\Lambda f \equiv 0 \equiv \partial_\Lambda f$, f is biconstant.*

6. Discrete differential forms

We mainly consider two type of functions, functions $f : V(\Lambda) \to \mathbb{C}$ and functions $h : V(\Diamond) \to \mathbb{C}$. An example for a relevant function on the quadrilateral faces is $\partial_\Lambda f$.

A *discrete one-form* ω is a complex function on the oriented edges of the medial graph X, and a *discrete two-form* Ω is a complex function on the faces of X. The evaluations of ω at an oriented edge e of X and of Ω at a face F of X are denoted by $\int_e \omega$ and $\iint_F \Omega$, respectively.

If P is a directed path of edges $e_1, e_2, \ldots, e_n$ of X, the *discrete integral* along P is defined as $\int_P \omega = \sum_{k=1}^n \int_{e_k} \omega$. For closed paths P, we write $\oint_P \omega$ instead. In the case that P is the boundary of an oriented disk in X, we call it a *discrete contour*. The *discrete integral* of Ω over several faces of X is defined similarly.

Definition 7. The discrete one-forms dz and $d\bar{z}$ are given by $\int_e dz = e$ and $\int_e d\bar{z} = \bar{e}$ for any oriented edge e of X. The discrete two-form Ω_0 is defined by

$$\iint_F \Omega_0 = -4i\text{area}(F).$$

Remark 8. Ω_0 is the straightforward discretization of $2dz \wedge d\bar{z}$. It turns out later that several discrete two-forms we are interested in are just defined on half of the faces of X and zero on the other elements of $F(X)$. In order to get results comparable to the classical theory after integration, a factor of two enters in the definitions of Sections 8 and 9. Introducing Ω_0 is a technical trick that allows us to implement this factor of two just in Ω_0. In local coordinates, we can perform our calculations with Ω_0 in the discrete setting exactly as we do with $dz \wedge d\bar{z}$ in the smooth theory, but integration of Ω_0 gives twice the value $dz \wedge d\bar{z}$ yields.

A discrete one-form ω is said to be of *type* $\Diamond$, if for any $Q \in V(\Diamond)$ there exist complex numbers p, q, such that $\omega = pdz + qd\bar{z}$ on all edges $e = [Q, v_s]$, $v_s \sim Q$.

Definition 9. Let $f : V(\Lambda) \to \mathbb{C}$, $h : V(\Diamond) \to \mathbb{C}$, ω a discrete one-form, and Ω a discrete two-form. For any edge $e = [Q, v]$ and any faces F_v, F_Q of X corresponding to the vertex star of $v \in V(\Lambda)$ or the Varignon parallelogram inside $Q \in V(\Diamond)$, we define the products $f\omega$, $h\omega$, $f\Omega$, and $h\Omega$ by

$$\int_e f\omega := f(v)\int_e \omega \quad \text{and} \quad \iint_{F_v} f\Omega := f(v)\iint_{F_v} \Omega, \quad \iint_{F_Q} f\Omega := 0;$$
$$\int_e h\omega := h(Q)\int_e \omega \quad \text{and} \quad \iint_{F_v} h\Omega := 0, \qquad \iint_{F_Q} h\Omega := h(Q)\iint_{F_Q} \Omega.$$

Lemma 10. *Let $Q \in V(\Diamond)$ and f be a complex function on the vertices of Q. Then,*

$$\partial_\Lambda f(Q) = \frac{-1}{4i\text{area}(F_Q)}\oint_{P_Q} fd\bar{z} \text{ and } \bar{\partial}_\Lambda f(Q) = \frac{1}{4i\text{area}(F_Q)}\oint_{P_Q} fdz.$$

Remark 11. The additional factor of $1/2$ is due to the fact that in analogy to the smooth setup, we should not multiply $f(v)$ with dz (or $d\bar{z}$), but by the arithmetic mean of $f(v)$ and some intermediate value $f(Q)$ instead. Integrating fdz would then eliminate $f(Q)$, so the choice of the intermediate value does not matter.

7. Discrete derivatives of functions on the faces of the quad-graph

Inspired by Lemma 10, we can now define the discrete derivatives for complex functions on $V(\Diamond)$. The reason for the additional factor of $1/2$ remains the same.

Definition 12. Let $v \in V(\Lambda)$ and h be a complex function defined on all quadrilaterals $Q_s \sim v$. Then, the *discrete derivatives* $\partial_\Diamond h$, $\bar{\partial}_\Diamond h$ at v are defined by

$$\partial_\Diamond h(v) := \frac{-1}{4i\text{area}(F_v)}\oint_{P_v} hd\bar{z} \text{ and } \bar{\partial}_\Diamond h(v) := \frac{1}{4i\text{area}(F_v)}\oint_{P_v} hdz.$$

h is called *discrete holomorphic* at v if $\bar{\partial}_\Diamond h(v) = 0$.

Note that in the rhombic case, our definition coincides with the one in [3]. As an immediate consequence of the definition, we obtain a *discrete Morera's theorem.*

Proposition 13. *$f : V(\Lambda) \to \mathbb{C}$ or $h : V(\Diamond) \to \mathbb{C}$ is discrete holomorphic if and only if $\oint_P fdz = 0$ or $\oint_P hdz = 0$, respectively, for all discrete contours P.*

Definition 14. Let $f_1, f_2 : V(\Lambda) \to \mathbb{C}$ and $h_1, h_2 : V(\Diamond) \to \mathbb{C}$. Their *discrete scalar products* are defined as

$$\langle f_1, f_2\rangle := -\frac{1}{2i}\iint_{F(X)} f_1\bar{f}_2\Omega_0 \quad \text{and} \quad \langle h_1, h_2\rangle := -\frac{1}{2i}\iint_{F(X)} h_1\bar{h}_2\Omega_0,$$

whenever the right hand side converges absolutely.

Note that both discrete two-forms $f_1\bar{f}_2\Omega_0$ and $h_1\bar{h}_2\Omega_0$ are zero on half of the faces of X, making the factor of two incorporated in Ω_0 necessary.

Proposition 15. *$-\partial_\Diamond$ and $-\bar{\partial}_\Diamond$ are the formal adjoints of $\bar{\partial}_\Lambda$ and ∂_Λ, respectively. That is, if $f : V(\Lambda) \to \mathbb{C}$ or $h : V(\Diamond) \to \mathbb{C}$ is compactly supported,*

$$\langle \partial_\Lambda f, h\rangle + \langle f, \bar{\partial}_\Diamond h\rangle = 0 = \langle \bar{\partial}_\Lambda f, h\rangle + \langle f, \partial_\Diamond h\rangle.$$

Proof. Using Lemma 10 and $\partial_\Diamond \bar{h} = \overline{\bar{\partial}_\Diamond h}$, we get

$$-2i\langle \partial_\Lambda f, h\rangle - 2i\langle f, \bar{\partial}_\Diamond h\rangle = \sum_{Q\in V(\Diamond)} \overline{h(Q)}\oint_{P_Q} fd\bar{z} + \sum_{v\in V(\Lambda)} f(v)\oint_{P_v} \bar{h}d\bar{z} = \oint_P f\bar{h}d\bar{z} = 0,$$

where P is a large contour such that $f\bar{h}$ vanishes in a neighborhood of P. The second equation is shown in the same way. ☐

Remark 16. In the work of Kenyon [12] and Mercat [15] on discrete complex analysis on rhombic quad-graphs, the discrete differentials for functions on the vertices and the faces were constructed in such a way that they are formal adjoints to each other.

As in the rhombic setup [3], the discrete differentials commute in the following way:

Proposition 17. *Let $f : V(\Lambda) \to \mathbb{C}$. Then, $\partial_\Diamond \bar{\partial}_\Lambda f \equiv \bar{\partial}_\Diamond \partial_\Lambda f$. In particular, $\partial_\Lambda f$ is discrete holomorphic if f is discrete holomorphic.*

Remark 18. Note that even in the rhombic case, $\partial_\Lambda \bar{\partial}_\Diamond h \neq \bar{\partial}_\Lambda \partial_\Diamond h$ for generic $h : V(\Diamond) \to \mathbb{C}$ [3].

Proposition 19. *Let $\Diamond_0 \subseteq \Diamond$ be simply-connected. Then, for any discrete holomorphic function h on $V(\Diamond_0)$, there is a discrete primitive $f := \int h$ on $V(\Lambda_0)$, i.e., f is discrete holomorphic and $\partial_\Lambda f = h$. f is unique up to two additive constants on Γ_0 and Γ_0^*.*

Proof. Since h is discrete holomorphic, $\oint_P hdz = 0$ for any discrete contour P. Thus, hdz can be integrated to a well-defined function f_X on $V(X)$ that is unique up to an additive constant. Using that hdz is a discrete one-form of type $\Diamond$, we can construct a function f on $V(\Lambda)$ such that $f_X\left((v+w)/2\right) = \left(f(v)+f(w)\right)/2$ for any edge (v,w) of Λ. Given f_X, f is unique up to an additive constant.

In summary, f is unique up to two additive constants that can be chosen independently on Γ_0 and Γ_0^*. By construction, f satisfies

$$\frac{f(b_+)-f(b_-)}{b_+-b_-} = h(Q) = \frac{f(w_+)-f(w_-)}{w_+-w_-}$$

on any quadrilateral Q. It follows that f is discrete holomorphic and $\partial_\Lambda f = h$. □

8. Discrete exterior derivative

Our notation of discrete exterior calculus is similar to the approach of Mercat in [14, 15, 16], but differs in some aspects. The main differences are due to our different notation of multiplication of functions with discrete one-forms, which allows us to define a discrete exterior derivative on a larger class of discrete one-forms. It coincides with Mercat's discrete exterior derivative in the case of discrete one-forms of type $\Diamond$. In contrast, our definitions are based on a coordinate representation.

Definition 20. Let $f : V(\Lambda) \to \mathbb{C}$ and $h : V(\Diamond) \to \mathbb{C}$. We define the *discrete exterior derivatives* df and dh as follows:

$$df := \partial_\Lambda f dz + \bar{\partial}_\Lambda f d\bar{z} \text{ and } dh := \partial_\Diamond h dz + \bar{\partial}_\Diamond h d\bar{z}.$$

Let ω be a discrete one-form. Around faces F_v and F_Q of X corresponding to vertices $v \in V(\Lambda)$ and $Q \in V(\Diamond)$, respectively, we write $\omega = pdz + qd\bar{z}$ with functions p, q defined on faces $Q_s \sim v$ or vertices $b_\pm, w_\pm \sim Q$, respectively. The *discrete exterior derivative* $d\omega$ is given by

$$d\omega|_{F_v} := \left(\partial_\Diamond q - \bar{\partial}_\Diamond p\right)\Omega_0 \text{ and } d\omega|_{F_Q} := \left(\partial_\Lambda q - \bar{\partial}_\Lambda p\right)\Omega_0.$$

The reason why we add a factor of two in the definition of $d\omega$ (hidden in Ω_0) is the same as the factor of $1/2$ in the definition of $\partial_\Diamond, \bar{\partial}_\Diamond$: For the definition of $d\omega$, p and q are defined on the vertices of Λ or $\Diamond$, but ω lives halfway between two incident vertices of Λ and $\Diamond$, resulting in the factor of 2.

The representation of ω as $pdz + qd\bar{z}$ (p, q defined on edges of X) is non-unique, since we represent one complex number as the linear combination of two other complex numbers. However, $d\omega$ is well-defined by *discrete Stokes' theorem*, which also justifies our definition of df and dh.

Lemma 21. *Let $f : V(\Lambda) \to \mathbb{C}$, and let ω be a discrete one-form. Then, for any directed edge e of X starting in the midpoint of the edge vv'_- and ending in the midpoint of the edge vv'_+ of Λ, and for any face F of X with counterclockwise oriented boundary ∂F we have:*

$$\int_e df = \frac{f(v)+f(v'_+)}{2} - \frac{f(v)+f(v'_-)}{2} \text{ and } \iint_F d\omega = \oint_{\partial F} \omega.$$

An easy consequence of the definition of the discrete exterior derivative is that $\iint_F d\omega = 0$ on any face F corresponding to a vertex of Λ, when ω is a discrete one-form of type $\Diamond$. We call a discrete one-form ω *closed*, if $d\omega \equiv 0$. For example, df is closed if f is a complex function on $V(\Lambda)$.

Proposition 22. *Let $f : V(\Lambda) \to \mathbb{C}$. Then, $ddf = 0$.*

Proof. By discrete Stokes' theorem, $ddf = 0$ if $\oint_P df = 0$ for any discrete elementary cycle P. Since df is of type $\Diamond$, the statement is trivially true if $P = P_Q$ for $Q \in V(\Diamond)$. So let $P = P_v$ for $v \in V(\Lambda)$. Using discrete Stokes' theorem again,

$$\oint\limits_{P_v} df = \sum_{Q_s \sim v} \frac{f(v_s') - f(v_{s-1}')}{2} = 0.$$

□

Remark 23. An analogous statement for functions $h : V(\Diamond) \to \mathbb{C}$ is not true in general, even if h is discrete holomorphic and Λ is a rhombic quad-graph.

Note that Proposition 22 immediately implies Proposition 17 by $ddf = \left(\partial_\Diamond \bar{\partial}_\Lambda f - \bar{\partial}_\Diamond \partial_\Lambda f\right) \Omega_0$.

Corollary 24. *Let $f : V(\Lambda) \to \mathbb{C}$. Then, f is discrete holomorphic if and only if $df = pdz$ is closed for some $p : V(\Diamond) \to \mathbb{C}$. In this case, p is discrete holomorphic.*

Corollary 25. *Let $f, g : V(\Lambda) \to \mathbb{C}$ and $h : V(\Diamond) \to \mathbb{C}$.*

(1) *$fdg + gdf$ is a closed discrete one-form.*
(2) *If f and h are discrete holomorphic, $fhdz$ is a closed discrete one-form.*

Proof. (1) Let $\omega := fdg + gdf$. By Proposition 22, df and dg are closed. Thus, $\oint_{\partial F} \omega = 0$ for any face F corresponding to $V(\Lambda)$. Using Lemma 10, a direct calculation shows $\oint_{\partial F} \omega = 0$ for any face F corresponding to $V(\Diamond)$. It follows by discrete Stokes' theorem that $d\omega = 0$.

(2) By discrete Morera's theorem, $\oint_{\partial F} fhdz = 0$ for any face F of X, so $fhdz$ is closed. □

Remark 26. In particular, a product $f \cdot g : V(X) \to \mathbb{C}$ can be defined by integration, and $f \cdot g$ is defined up to an additive constant. Furthermore, $f \cdot h : E(X) \to \mathbb{C}$ can be defined by "pointwise" multiplication. If all these functions are holomorphic, $fdg + gdf = pdz$ is closed ($p : E(X) \to \mathbb{C}$) and so to say a discrete holomorphic one-form, meaning that $f \cdot g$ is discrete holomorphic in this sense. Similarly, $fhdz$ is closed, so $f \cdot h$ is kind of discrete holomorphic by a discrete Morera's theorem. However, $f \cdot g$ and $f \cdot h$ are generally not discrete holomorphic everywhere according to the classical quad-based definition of discrete holomorphicity on the dual of a bipartite quad-graph [9].

9. Discrete wedge product

Following Whitney [19], Mercat defined in [14] a discrete wedge product for discrete one-forms living on the edges of Λ. Then, the discrete exterior derivative defined by a discretization of Stokes' theorem is a derivation for the discrete wedge product. However, a discrete Hodge star cannot be defined on Λ. To circumvent this problem, Mercat used an averaging map to relate discrete one-forms on the edges of Λ with discrete one-forms on the edges of Γ and Γ^*, i.e., discrete one-forms of type $\Diamond$. Then, he could define a discrete Hodge star; however, the discrete exterior derivative was not a derivation for the now heteregoneous discrete wedge product anymore.

We propose a different interpretation of the discrete wedge product. It the end, we somehow recover the definitions Mercat proposed in [14, 15, 16], but our derivation is different. Starting with discrete one-forms of type $\Diamond$ that are defined on the edges of X, we obtain a discrete wedge product on the faces of X that vanishes on half of the faces. This definition is different from Whitney's [19] and has the advantage that both a discrete wedge product and a discrete Hodge star can be defined on the same structure. In contrast to Mercat's work, we now can make sense out of the statement that the discrete exterior derivative is a derivation for the discrete wedge product, see Proposition 29. This proposition is of crucial importance to deduce discrete integral formulae such as discrete Green's identities.

Lemma 27. *Let ω be a discrete one-form of type $\Diamond$. Then, there is a unique representation $\omega = pdz + qd\bar{z}$ with functions $p, q : V(\Diamond) \to \mathbb{C}$. On a quadrilateral $Q \in V(\Diamond)$, p and q are given by*

$$p(Q) = \frac{1}{2\sin(\varphi_Q)} \left(\exp\left(-i\left(\varphi_Q - \frac{\pi}{2}\right)\right) \frac{\int_e \omega}{e} + \exp\left(i\left(\varphi_Q - \frac{\pi}{2}\right)\right) \frac{\int_{e^*} \omega}{e^*} \right),$$
$$q(Q) = \frac{1}{2\sin(\varphi_Q)} \left(\exp\left(i\left(\varphi_Q - \frac{\pi}{2}\right)\right) \frac{\int_e \omega}{\bar{e}} + \exp\left(-i\left(\varphi_Q - \frac{\pi}{2}\right)\right) \frac{\int_{e^*} \omega}{\bar{e}^*} \right).$$

Here, e is an edge of X parallel to a (black) edge of Γ, and e^* corresponds to an (white) edge of Γ^*.

Definition 28. Let $\omega = pdz + qd\bar{z}$ and $\omega' = p'dz + q'd\bar{z}$ be two discrete one-forms of type $\Diamond$, $p, p', q, q' : V(\Diamond) \to \mathbb{C}$ given by Lemma 27. Then, the *discrete wedge product* $\omega \wedge \omega'$ is defined as the discrete two-form being 0 on faces of X corresponding to vertices of Λ that equals

$$(pq' - qp')\,\Omega_0$$

on faces corresponding to $V(\Diamond)$.

By definition, the discrete wedge product vanishes on faces of X corresponding to $V(\Lambda)$. Since the faces of X corresponding to $V(\Diamond)$ cover exactly half of the area of the quadrilaterals, the factor of two in the definition of Ω_0 compared to $dz \wedge d\bar{z}$ incorporates the vanishing regions of the discrete wedge product.

Proposition 29. *If $f : V(\Lambda) \to \mathbb{C}$ and ω is a discrete one-form of type $\Diamond$, $d(f\omega) = df \wedge \omega + fd\omega$.*

Proof. Let $\omega = pdz + qd\bar{z}$ with $p, q : V(\Diamond) \to \mathbb{C}$ given by Lemma 27. For $v \in V(\Lambda)$ and $Q \in V(\Diamond)$,

$$\begin{aligned} d(f\omega)|_{F_v} &= \left(f(v)\,(\partial_\Diamond q)\,(v) - f(v)\left(\bar{\partial}_\Diamond p\right)(v)\right)\Omega_0 = fd\omega|_{F_v}, \\ d(f\omega)|_{F_Q} &= \left(q(Q)\,(\partial_\Lambda f)\,(Q) - p(Q)\left(\bar{\partial}_\Lambda f\right)(Q)\right)\Omega_0 = (df \wedge \omega))|_{F_Q}. \end{aligned}$$

But $(df \wedge \omega)|_{F_v} = 0$ and $fd\omega|_{F_Q} = 0$, so $d(f\omega) = df \wedge \omega + fd\omega$. □

10. DISCRETE HODGE STAR

Definition 30. Let $f : F(\Lambda) \to \mathbb{C}$, $h : V(\Diamond) \to \mathbb{C}$, $\omega = pdz + qd\bar{z}$ a discrete one-form of type $\Diamond$ with complex functions $p, q : V(\Diamond) \to \mathbb{C}$ given by Lemma 27, and Ω a discrete two-form. The *discrete Hodge star* is given by

$$\star f := -\frac{1}{2i} f\Omega_0; \quad \star h := -\frac{1}{2i} h\Omega_0; \quad \star\omega := -ipdz + iqd\bar{z}; \quad \star\Omega := -2i\frac{\Omega}{\Omega_0}.$$

If ω and ω' are both discrete one-forms of type $\Diamond$, we define their *discrete scalar product*

$$\langle \omega, \omega' \rangle := \iint\limits_{F(X)} \omega \wedge \star\bar{\omega}',$$

whenever the right hand side converges absolutely. Similarly, a discrete scalar product for discrete two-forms is defined.

Note that $\star\Omega$ is a priori a function on $F(X)$. However, the discrete two-forms to that we will apply the discrete Hodge star vanish on all faces of X corresponding to faces of Λ or on all faces corresponding to vertices of Λ. In these cases, $\star\Omega$ is a function on $V(\Lambda)$ or on $V(\Diamond)$, respectively.

Corollary 31.
(1) $\star^2 = \mathrm{Id}$ *on complex functions on $V(\Lambda)$ or $V(\Diamond)$ and discrete two-forms.*
(2) $\star^2 = -\mathrm{Id}$ *on discrete one-forms of type $\Diamond$.*
(3) $\langle f_1, f_2 \rangle = \iint_{F(X)} f_1 \overline{\star f_2}$ *and* $\langle h_1, h_2 \rangle = \iint_{F(X)} h_1 \overline{\star h_2}$ *for functions $f_1, f_2 : V(\Lambda) \to \mathbb{C}$ and $h_1, h_2 : V(\Diamond) \to \mathbb{C}$.*
(4) *$f : V(\Lambda) \to \mathbb{C}$ is discrete holomorphic if and only if $\star df = -idf$.*

Remark 32. It can be easily checked that our definition of a discrete Hodge star on discrete one-forms coincides with Mercat's definition given in [16]. But on discrete two-forms and complex functions, our definition of the discrete Hodge star includes an additional factor of the area of the corresponding face of X. As before, the additional factor of two encoded in Ω_0 reflects the fact that the corresponding two-forms vanish on half of the faces of X.

Proposition 33. *$\delta := -\star d \star$ is the formal adjoint of the discrete exterior derivative d: Let $f : V(\Lambda) \to \mathbb{C}$, ω a discrete one-form of type $\Diamond$, and Ω a discrete two-form being 0 on all faces corresponding to vertices of $\Diamond$. Assume that all of them are compactly supported. Then,*

$$\langle df, \omega \rangle = \langle f, \delta\omega \rangle \text{ and } \langle d\omega, \Omega \rangle = \langle \omega, \delta\Omega \rangle.$$

Proof. Using discrete Stokes' theorem, Proposition 29, and Corollary 31 (i), we obtain

$$0 = \iint\limits_{F(X)} d(f \star \bar{\omega}) = \iint\limits_{F(X)} df \wedge \star\bar{\omega} + \iint\limits_{F(X)} f d \star \bar{\omega} = \langle df, \omega \rangle + \langle f, \star d \star \omega \rangle.$$

The second equation is shown in the same manner. ☐

11. Discrete Laplacian

The discrete Laplacian and the discrete Dirichlet energy on general quad-graphs were first introduced by Mercat in [16]. Later, Skopenkov reintroduced these definitions in [18], taking the same definition in a different notation.

Definition 34. The *discrete Laplacian* on discrete differential forms is defined as the operator

$$\triangle := -\delta d - d\delta = \star d \star d + d \star d \star .$$

A function $f : V(\Lambda) \to \mathbb{C}$ is called *discrete harmonic* at $v \in V(\Lambda)$ if $\triangle f(v) = 0$.

The following factorization of the discrete Laplacian in terms of discrete derivatives generalizes the corresponding results given in [3] to general quad-graphs. The local representation of $\triangle f$ at $v \in V(\Lambda)$ is, up to a factor involving the area of the face F_v, the same as in [16].

Corollary 35. *Let $f : V(\Lambda) \to \mathbb{C}$. Then, $\triangle f = 4\partial_\Diamond \bar{\partial}_\Lambda f = 4\bar{\partial}_\Diamond \partial_\Lambda f$. At a vertex v of Λ,*

$$\triangle f(v) = \frac{1}{4\mathrm{area}(F_v)} \sum_{Q_s \sim v} \frac{1}{\mathrm{Re}\,(\rho(v,v_s))} \left(|\rho(v,v_s)|^2 \left(f(v_s) - f(v)\right) + \mathrm{Im}\,(\rho(v,v_s)) \left(f(v'_s) - f(v'_{s-1})\right)\right).$$

Remark 36. In the case that the diagonals of the quadrilaterals are orthogonal to each other, ρ is always real. Then, the discrete Laplacian splits into two discrete Laplacians on Γ and Γ^*.

Corollary 37. *Let $f : V(\Lambda) \to \mathbb{C}$.*

(1) *If f is discrete harmonic, $\partial_\Lambda f$ is discrete holomorphic.*
(2) *If f is discrete holomorphic, f, $\mathrm{Re}\, f$, and $\mathrm{Im}\, f$ are discrete harmonic.*

For a finite subset $\Diamond_0 \subset \Diamond$ and two functions $f, g : V(\Lambda_0) \to \mathbb{C}$, we denote by

$$\langle f, g \rangle_{\Diamond_0} := -\frac{1}{2i} \iint\limits_{F(X_0)} f \bar{g} \Omega_0$$

the discrete scalar product of f and g restricted to $\Diamond_0$. Similarly, the restriction of the discrete scalar product of two discrete one-forms is defined.

In the rhombic setup, discrete versions of Green's second identity were already stated by Mercat [14], whose integrals were not well defined separately, and Chelkak and Smirnov [3], whose boundary integral was an explicit sum involving boundary angles. We are able to provide a *discrete Green's first identity* out of which *discrete Green's second identity* immediately follows. The formulation and the proof is a complete analog to the smooth setting.

Theorem 38. *Let $\Diamond_0 \subset \Diamond$ be finite, and let $f, g : V(\Lambda_0) \to \mathbb{C}$.*

(1) $\langle f, \triangle g \rangle_{\Diamond_0} + \langle df, dg \rangle_{\Diamond_0} = \oint\limits_{\partial X_0} f \star d\bar{g}.$
(2) $\langle \triangle f, g \rangle_{\Diamond_0} - \langle f, \triangle g \rangle_{\Diamond_0} = \oint\limits_{\partial X_0} \left(f \star d\bar{g} - \bar{g} \star df\right).$

Proof. By Proposition 29, $d\left(f \star d\bar{g}\right) = df \wedge \star d\bar{g} + f \star \left(\star d \star d\bar{g}\right)$. Now, discrete Stokes' theorem yields the desired result. For the second part, just apply twice discrete Green's first identity. ☐

12. Discrete Cauchy's integral formulae

Definition 39. Functions $K_{Q_0} : V(\Lambda) \to \mathbb{C}$ and $K_{v_0} : V(\Diamond) \to \mathbb{C}$ are called *discrete Cauchy's kernels* (*with respect to* $Q_0 \in V(\Diamond)$ or $v_0 \in V(\Lambda)$, respectively), if for all $Q \in V(\Diamond), v \in V(\Lambda)$ there holds:

$$\bar{\partial}_\Lambda K_{Q_0}(Q) = \delta_{QQ_0} \frac{\pi}{2\mathrm{area}(F_Q)} \text{ and } \bar{\partial}_\Diamond K_{v_0}(v) = \delta_{vv_0} \frac{\pi}{2\mathrm{area}(F_v)}.$$

Remark 40. In the general case, it seems to be practically impossible to speak about any asymptotic behavior of certain functions, as Kenyon did for discrete Green's functions and discrete Cauchy's kernels on rhombic quad-graphs [12]. For this reason, we do not require any asymptotic behavior of discrete Cauchy's kernels. However, we can construct discrete Cauchy's kernels on parallelogram graphs with appropriate asymptotics and can prove at least existence of discrete Cauchy's kernels with respect to $Q_0 \in V(\Diamond)$ or $v_0 \in V(\Lambda)$ in the general case [9].

Theorem 41. *Let f and h be discrete holomorphic functions on $V(\Lambda)$ and $V(\Diamond)$, respectively. Let $v_0 \in V(\Lambda)$ and $Q_0 \in V(\Diamond)$, and let $K_{v_0} : V(\Diamond) \to \mathbb{C}$ and $K_{Q_0} : V(\Lambda) \to \mathbb{C}$ be discrete Cauchy's kernels with respect to v_0 and Q_0, respectively.*

Then, for any discrete contours C_{v_0} and C_{Q_0} on X surrounding v_0 and Q_0 once in counterclockwise order, respectively, discrete Cauchy's integral formulae are true:

$$f(v_0) = \frac{1}{2\pi i} \oint\limits_{C_{v_0}} f K_{v_0} dz \text{ and } h(Q_0) = \frac{1}{2\pi i} \oint\limits_{C_{Q_0}} h K_{Q_0} dz.$$

Remark 42. In the case of rhombic quad-graphs, Mercat formulated a discrete Cauchy's integral formula for the average of a discrete holomorphic function on $V(\Lambda)$ along an edge of Λ. In [3], Chelkak and Smirnov provided a discrete Cauchy's integral formula for discrete holomorphic functions on $V(\Diamond)$ using two integrals along cycles on Γ and Γ^*.

Theorem 43. *Let $f : V(\Lambda) \to \mathbb{C}$ be discrete holomorphic, $Q_0 \in V(\Diamond)$, and let $K_{Q_0} : V(\Lambda) \to \mathbb{C}$ be a discrete Cauchy's kernel with respect to Q_0.*

Then, for any discrete contour C_{Q_0} in X surrounding Q_0 once in counterclockwise order that does not contain any edge inside Q_0, the discrete Cauchy's integral formula is true:

$$\partial_\Lambda f(Q_0) = -\frac{1}{2\pi i} \oint\limits_{C_{Q_0}} f \partial_\Lambda K_{Q_0} dz.$$

Proof. Let D be the discrete domain in X bounded by C_{Q_0}. Since no edge of C_{Q_0} passes through Q_0, the discrete one-form $\bar{\partial} K_{Q_0} d\bar{z}$ vanishes on C_{Q_0}. Therefore,

$$\oint\limits_{C_{Q_0}} f \partial_\Lambda K_{Q_0} dz = \oint\limits_{C_{Q_0}} f dK_{Q_0} = \iint\limits_{D} d(f dK_{Q_0}) = \iint\limits_{D} df \wedge dK_{Q_0}$$

due to discrete Stokes' theorem, and Propositions 22 and 29. Now, f is discrete holomorphic, so $df \wedge dK_{Q_0} = \partial_\Lambda f \bar{\partial}_\Lambda K_{Q_0} \Omega_0$. But $\bar{\partial}_\Lambda K_{Q_0}$ vanishes on all vertices of $\Diamond$ but Q_0. Finally,

$$-\frac{1}{2\pi i} \oint\limits_{C_{Q_0}} f \partial_\Lambda K_{Q_0} dz = -\frac{1}{2\pi i} \iint\limits_{F_{Q_0}} \partial_\Lambda f \bar{\partial}_\Lambda K_{Q_0} \Omega_0 = \partial_\Lambda f(Q_0).$$

□

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