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Birings and plethories of integer-valued polynomials

Jesse ELLIOTT
Abstract

Let A and B be commutative rings with identity. An A - B -biring is an A -algebra S together with a lift of the functor $\text{Hom}_A(S, -)$ from A -algebras to sets to a functor from A -algebras to B -algebras. An A -plethory is a monoid object in the monoidal category, equipped with the composition product, of A - A -birings. The polynomial ring $A[X]$ is an initial object in the category of such structures. The D -algebra $\text{Int}(D)$ has such a structure if $D = A$ is a domain such that the natural D -algebra homomorphism $\theta_n : \bigotimes_{D, i=1}^n \text{Int}(D) \rightarrow \text{Int}(D^n)$ is an isomorphism for $n = 2$ and injective for $n \leq 4$. This holds in particular if θ_n is an isomorphism for all n , which in turn holds, for example, if D is a Krull domain or more generally a TV PVMD. In these cases we also examine properties of the functor $\text{Hom}_D(\text{Int}(D), -)$ from D -algebras to D -algebras, which we hope to show is a new object worthy of investigation in the theory of integer-valued polynomials.

1. INTRODUCTION

This paper is a summary of the results contained in the forthcoming paper [7]. Throughout this paper all rings and algebras are assumed commutative with identity.

For any integral domain D with quotient field K , any set \mathbf{X} , and any subset \mathbf{E} of $K^{\mathbf{X}}$, the ring of *integer-valued polynomials on \mathbf{E} over D* is the subring

$$\text{Int}(\mathbf{E}, D) = \{f(\mathbf{X}) \in K[\mathbf{X}] : f(\mathbf{E}) \subseteq D\}$$

of the polynomial ring $K[\mathbf{X}]$. In other words, $\text{Int}(\mathbf{E}, D)$ is the pullback of the direct product $D^{\mathbf{E}}$ along the K -algebra homomorphism $K[\mathbf{X}] \rightarrow K^{\mathbf{E}}$ acting by $f \mapsto (f(\underline{a}))_{\underline{a} \in \mathbf{E}}$. One writes $\text{Int}(D^{\mathbf{X}}) = \text{Int}(D^{\mathbf{X}}, D)$ and $\text{Int}(D) = \text{Int}(D, D)$. One also writes $\text{Int}(D^n) = \text{Int}(D^{\mathbf{X}})$ if \mathbf{X} is a set of cardinality n .

Much of the theory of integer-valued polynomial rings developed in attempts to generalize results known about $\text{Int}(\mathbb{Z})$ to $\text{Int}(D)$. This paper is concerned with finding such a generalization of a particular result about $\text{Int}(\mathbb{Z})$. To state this result we need a few definitions.

A ring A is said to be *binomial* if A is \mathbb{Z} -torsion-free and $\frac{a(a-1)(a-2)\cdots(a-n+1)}{n!} \in A \otimes_{\mathbb{Z}} \mathbb{Q}$ lies in A for all $a \in A$ and all positive integers n . For any set \mathbf{X} the ring $\text{Int}(\mathbb{Z}^{\mathbf{X}})$ is the free binomial ring generated by \mathbf{X} , and a \mathbb{Z} -torsion-free ring A is binomial if and only if, for every $a \in A$, there exists a ring homomorphism $\text{Int}(\mathbb{Z}) \rightarrow A$ sending X to a [4]. By the category of binomial rings we will mean the full subcategory of the category of rings whose objects are the binomial rings. By [1, Section 46] and [4, Theorem 9.1] we have the following.

Proposition 1. *There is a functor Bin from rings to binomial rings that is left-represented by $\text{Int}(\mathbb{Z})$ and is a right adjoint for the inclusion from binomial rings to rings.*

Our motivating problem is to generalize the above result to $\text{Int}(D)$ for further domains D . More specifically, we are interested in the following.

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Problem 2. *Determine all domains D for which $\text{Int}(D)$ left-represents a right adjoint for the inclusion from \mathcal{C} to the category of D -algebras for some full subcategory \mathcal{C} of the category of D -algebras.*

In particular, if D is such a domain, then the functor $\text{Hom}_D(\text{Int}(D), -)$ from D -algebras to sets must lift to a functor from D -algebras to D -algebras in \mathcal{C} . If $D = \mathbb{Z}$, then by Proposition 1 this holds for the category \mathcal{C} of binomial rings. Given a domain D , a natural candidate for the category \mathcal{C} is the category of D -torsion-free D -algebras that are “weakly polynomially complete” [5, Section 7], where a D -algebra A is said to be *weakly polynomially complete*, or *WPC*, if for every $a \in A$ there exists a D -algebra homomorphism $\text{Int}(D) \rightarrow A$ sending X to a . A binomial ring is equivalently a \mathbb{Z} -torsion-free WPC \mathbb{Z} -algebra, and for any domain D the D -algebra $\text{Int}(D)$ is itself WPC. Our goal, then, is to construct a right adjoint for the inclusion from the category of D -torsion-free WPC D -algebras to the category of D -algebras that is left-represented by $\text{Int}(D)$. In our efforts to do so we found it necessary to utilize the notions of a *biring* and a *plethory*.

Let A and B be rings. An A - B -*biring* is an A -algebra S together with the structure on S of a B -algebra object in the opposite category of the category of A -algebras. Thus an A - B -biring is an A -algebra S equipped with two *binary co-operations* $S \mapsto S \otimes_A S$, called *co-addition* and *co-multiplication* (both of which are A -algebra homomorphisms), along with a *co- B -linear structure* $B \rightarrow \text{Hom}_A(S, A)$, satisfying laws dual to those defining the A -algebras. By Yoneda’s lemma, an A - B -biring is equivalently an A -algebra S together with a lift of the covariant functor $\text{Hom}_A(S, -)$ it represents to a functor from the category of A -algebras to the category of B -algebras. (See any of [1, 2, 9] for the details.) For example, the polynomial ring $A[X]$ is an A - A -biring as it represents the identity functor from the category of A -algebras to itself. Co-addition acts by $X \mapsto X \otimes 1 + 1 \otimes X$, co-multiplication by $X \mapsto X \otimes X$, and the co-linear structure by $a \mapsto (f \mapsto f(a))$.

Proposition 3. *Let D be an integral domain.*

- (1) *The existence of a D - D -biring structure on $\text{Int}(D)$ is equivalent to the existence of a lift of the functor $\text{Hom}_D(\text{Int}(D), -)$ from D -algebras to sets to a functor from D -algebras to D -algebras.*
- (2) *A D - D -biring structure on $\text{Int}(D)$ is compatible with the D - D -biring structure on $D[X]$, that is, the inclusion $D[X] \rightarrow \text{Int}(D)$ is a homomorphism of D - D -birings, if and only if the natural map $\text{Hom}_D(\text{Int}(D), A) \rightarrow A$ given by $\varphi \mapsto \varphi(X)$ is a D -algebra homomorphism for every D -algebra A .*

Consequently, any solution to Problem 2 would yield conditions on integral domains D under which the D -algebra $\text{Int}(D)$ has a D - D -biring structure. Regarding the latter problem we have the following.

Theorem 4. *Assume that $\text{Int}(D)$ is flat over D , or more generally that the n -th tensor power $\text{Int}(D)^{\otimes n}$ of $\text{Int}(D)$ over D is D -torsion-free for $n \leq 4$. Then the domain $\text{Int}(D)$ has a (necessarily unique) D - D -biring structure that is compatible with the D - D -biring structure on $D[X]$ if and only if for every $f \in \text{Int}(D)$ the polynomials $f(X + Y)$ and $f(XY)$ both can be expressed as sums of polynomials of the form $g(X)h(Y)$ for $g, h \in \text{Int}(D)$.*

In analogy with ordinary polynomial rings, there is for any set \mathbf{X} a canonical D -algebra homomorphism

$$\theta_{\mathbf{X}} : \bigotimes_{X \in \mathbf{X}} \text{Int}(D) \longrightarrow \text{Int}(D^{\mathbf{X}}),$$

where the (possibly infinite) tensor product is over D and is a coproduct in the category of D -algebras. However, we do not know whether or not $\theta_{\mathbf{X}}$ is an isomorphism for every domain D and every set \mathbf{X} . There are several large classes of domains for which $\theta_{\mathbf{X}}$ is an isomorphism for all \mathbf{X} , such as the Krull domains, the almost Newtonian domains [5, Section 5], and the PVMDs D such that $\text{Int}(D_{\mathfrak{m}}) = \text{Int}(D)_{\mathfrak{m}}$ for every maximal ideal \mathfrak{m} , hence the TV PVMDs as well. (See [8] for the definition of a PVMD and a TV PVMD.) We say that the domain D is *polynomially composite* if $\theta_{\mathbf{X}}$ is an isomorphism for every set \mathbf{X} . Section 4 of [6] collects several known classes of polynomial composite domains.

Corollary 5. *If D is a polynomially composite domain, and in particular if D is a Krull domain or TV PVMD, then $\text{Int}(D)$ has a unique D - D -biring structure such that the inclusion $D[X] \rightarrow \text{Int}(D)$ is a homomorphism of D - D -birings.*

By [4, Proposition 9.3] one has $\text{Bin}(A) \cong \mathbb{Z}_p$ for any integral domain A of characteristic p , where \mathbb{Z}_p denotes the ring of p -adic integers, and in particular one has $\text{Bin}(\mathbb{F}_p) \cong \mathbb{Z}_p$. This generalizes as follows.

Proposition 6. *Let D be a Dedekind domain, and let \mathfrak{m} be a maximal ideal of D with finite residue field. Then the map*

$$\widehat{D}_{\mathfrak{m}} \longrightarrow \text{Hom}_D(\text{Int}(D), D/\mathfrak{m})$$

acting by $\alpha \mapsto (f \mapsto f(\alpha) \bmod \mathfrak{m}\widehat{D}_{\mathfrak{m}})$ is a D -algebra isomorphism. More generally, for any domain extension A of D with $\mathfrak{m}A = 0$, the diagram

$$\begin{array}{ccc} \widehat{D}_{\mathfrak{m}} & \longrightarrow & \text{Hom}_D(\text{Int}(D), D/\mathfrak{m}) \\ & \searrow & \downarrow \\ & & \text{Hom}_D(\text{Int}(D), A) \end{array}$$

is a commutative diagram of D -algebra isomorphisms.

By [2, Proposition 1.4], for any A - B -biring S , the lifted functor $\text{Hom}_A(S, -)$ from A -algebras to B -algebras has a left adjoint, denoted $S \odot_A -$. In analogy with the tensor product, the A -algebra $S \odot_A R$ for any B -algebra R is the A -algebra generated by the symbols $s \odot r$ for all $s \in S$ and $r \in R$, subject to the relations [2, 1.3.1-2]. If S and T are A - A -birings, then so is $S \odot_A T$, and the category of A - A -birings equipped with the operation \odot_A is monoidal with unit $A[X]$. An A -plethory is a monoid object in that monoidal category, that is, it is an A - A -biring P together with an associative map $\circ : P \odot_A P \rightarrow P$ of A - A -birings (called *composition*) possessing a unit $e : A[X] \rightarrow P$. (See any of [1, 2, 9] for details on these constructions.) An A -plethory is also known as an A - A -biring monad object, an A - A -biring triple, or a Tall-Wraith monad object in the category of A -algebras. For example, for any ring A , the polynomial ring $A[X]$ has the structure of an A -plethory and in fact is an initial object in the category of A -plethories.

Proposition 7. *Let D be an integral domain. Any D - D -biring structure on $\text{Int}(D)$ compatible with that on $D[X]$ extends uniquely to a D -plethory structure on $\text{Int}(D)$ with unit given by the inclusion $D[X] \rightarrow \text{Int}(D)$. Composition $\circ : \text{Int}(D) \odot_D \text{Int}(D) \rightarrow \text{Int}(D)$ acts by ordinary composition on elements of the form $f \odot g$, that is, one has $\circ : f \odot g \mapsto f \circ g$ for all $f, g \in \text{Int}(D)$.*

The following theorem gives a partial solution to Problem 2.

Theorem 8. *Let D be an integral domain.*

- (1) *Assume that there exists a D - D -biring structure on $\text{Int}(D)$ compatible with that on $D[X]$. Then the functors $\text{Hom}_D(\text{Int}(D), -)$ and $\text{Int}(D) \odot_D -$ are right and left adjoints, respectively, of the inclusion from D -torsion-free WPC algebras to D -algebras if and only if the D -algebras $\text{Hom}_D(\text{Int}(D), A)$ and $\text{Int}(D) \odot_D A$ are D -torsion-free for any D -algebra A .*
- (2) *If D is a PID with all residue fields finite, then the hypotheses (and therefore the conclusion) of statement (1) hold.*

Note that if $\text{Int}(D) = D[X]$, which, for example, holds by [3, Corollary I.3.7] if D has no finite residue fields, then $\text{Hom}_D(\text{Int}(D), A)$ is naturally isomorphic to A and in particular is not D -torsion-free if A is not D -torsion-free. Of course in that case every D -algebra is WPC.

2. WPC D -ALGEBRAS AND TENSOR POWERS OF $\text{Int}(D)$

As in [5, Section 7] and as in the introduction, we will say that a D -algebra A is *weakly polynomially complete*, or *WPC*, if for every $a \in A$ there exists a D -algebra homomorphism $\text{Int}(D) \rightarrow A$ sending X to a . A D -torsion-free D -algebra A is WPC if and only if $f(A) \subseteq A$ for all $f \in \text{Int}(D) \subseteq (A \otimes_D K)[X]$. In particular, a domain extension A of D is WPC if and only if $\text{Int}(D) \subseteq \text{Int}(A)$.

For any set \mathbf{X} , the smallest subring of $\text{Int}(D^{\mathbf{X}})$ containing $D[\mathbf{X}]$ that is closed under precomposition by elements of $\text{Int}(D)$ is denoted $\text{Int}_w(D^{\mathbf{X}})$. For any domain D (finite or infinite), the domain $\text{Int}_w(D^{\mathbf{X}})$ is the free WPC extension of D generated by \mathbf{X} [5, Proposition 7.2]. It is also the *weak polynomial completion* $w_D(D[\mathbf{X}])$ of $D[\mathbf{X}]$ with respect to D , as defined in [5, Section 8] and in Proposition 11 below.

If $\text{Int}_{\otimes}(D^{\mathbf{X}})$ denotes the image of the D -algebra homomorphism $\theta_{\mathbf{X}} : \bigotimes_{X \in \mathbf{X}} \text{Int}(D) \rightarrow \text{Int}(D^{\mathbf{X}})$, then we have $\text{Int}_{\otimes}(D^{\mathbf{X}}) \subseteq \text{Int}_w(D^{\mathbf{X}})$, and equality holds for a given set \mathbf{X} if and only if $\text{Int}_{\otimes}(D^{\mathbf{X}})$ is a WPC extension of D . If equality holds for any set \mathbf{X} then we will say that D is *weakly polynomially composite*.

Proposition 9. *The following conditions are equivalent for any integral domain D .*

- (1) D is weakly polynomially composite.
- (2) $\text{Int}_{\otimes}(D^{\mathbf{X}})$ is a WPC extension of D for any set \mathbf{X} .
- (3) $\text{Int}_{\otimes}(D^n)$ is a WPC extension of D for some integer $n > 1$.
- (4) $\text{Int}_{\otimes}(D^2)$ is a WPC extension of D .
- (5) For any element f of $\text{Int}(D)$, the polynomials $f(X + Y)$ and $f(XY)$ lie in the image of $\theta_{\{X, Y\}}$.
- (6) The compositum of any collection of WPC D -algebras of D contained in some D -torsion-free D -algebra is again a WPC D -algebra.
- (7) The compositum of any collection of WPC extensions of D contained in some domain extension of D is again a WPC extension of D .

Clearly polynomial compositeness implies weak polynomial compositeness.

At the end of Section 8 of [5] it is noted how to construct the left adjoint of the inclusion functor from WPC domain extensions of D to domain extensions of D . The proof can be easily generalized to establish the following.

Proposition 10. *Let D be a domain with quotient field K , and let A be a D -torsion-free D -algebra.*

- (1) A is contained in a smallest D -torsion-free WPC D -algebra, denoted $w_D(A)$, equal to the intersection of all WPC D -algebras containing A and contained in $A \otimes_D K$.
- (2) One has $w_D(A) = A$ if and only if A is WPC, and $w_D(A)$ is a domain if and only if A is a domain.
- (3) One has $w_D(A) \cong \text{Int}_w(D^{\mathbf{X}}) / ((\ker \varphi)K \cap \text{Int}_w(D^{\mathbf{X}}))$ for any surjective D -algebra homomorphism $\varphi : D[\mathbf{X}] \rightarrow A$.
- (4) The association $A \mapsto w_D(A)$ defines a functor from the category of D -torsion-free D -algebras to the category of D -torsion-free WPC D -algebras—both categories with morphisms as D -algebra homomorphisms—that is a left adjoint for the inclusion functor.

Assuming that D is weakly polynomially composite, we can also construct the right adjoint of the inclusion functor from D -torsion-free WPC D -algebras to D -torsion-free D -algebras.

Proposition 11. *Let D be a weakly polynomially composite domain, and let A be a D -torsion-free D -algebra.*

- (1) A contains a largest WPC D -algebra, denoted $w^D(A)$, equal to the compositum of all WPC D -algebras contained in A .
- (2) One has $w^D(A) = A$ if and only if A is WPC.
- (3) One has $w^D(A) = \{a \in A : a = \varphi(X) \text{ for some } \varphi \in \text{Hom}_D(\text{Int}(D), A)\}$.
- (4) The association $A \mapsto w^D(A)$ defines a functor from the category of D -torsion-free D -algebras to the category of D -torsion-free WPC D -algebras—both categories with morphisms as D -algebra homomorphisms—that is a right adjoint for the inclusion functor.

3. BIRING AND PLETHORY STRUCTURE ON $\text{Int}(D)$

The following result implies Theorem 4 and Corollary 5 of the introduction.

Theorem 12. *Let D be an integral domain.*

- (1) If the domain $\text{Int}(D)$ has a D - D -biring structure such that the inclusion $D[X] \rightarrow \text{Int}(D)$ is a homomorphism of D - D -birings, then D is weakly polynomially composite.

- (2) Assume that the n -th tensor power $\text{Int}(D)^{\otimes n}$ of $\text{Int}(D)$ over D is D -torsion-free for $n \leq 4$. Then $\text{Int}(D)$ has a unique D - D -biring structure such that the inclusion $D[X] \rightarrow \text{Int}(D)$ is a homomorphism of D - D -birings if D is weakly polynomially composite.

The plethory $A[X]$ is an initial object in the category of A -plethories. Like $A[X]$, and in particular like the domain $D[X]$, the domain $\text{Int}(D)$ has its own “internal” operation of composition. This leads to the following result.

Proposition 13. *Let D be an integral domain. Any D - D -biring structure on $\text{Int}(D)$ such that the inclusion $D[X] \rightarrow \text{Int}(D)$ is a homomorphism of D - D -birings extends uniquely to a D -plethory structure on $\text{Int}(D)$ with unit given by the inclusion $D[X] \rightarrow \text{Int}(D)$. Composition $\circ : \text{Int}(D) \odot_D \text{Int}(D) \rightarrow \text{Int}(D)$ acts by ordinary composition on elements of the form $f \odot g$, that is, one has $\circ : f \odot g \mapsto f \circ g$ for all $f, g \in \text{Int}(D)$.*

Corollary 14. *If D is a polynomially composite domain, and in particular if D is a Krull domain or TV PVMD, then $\text{Int}(D)$ has a unique D -plethory structure with unit given by the inclusion $D[X] \rightarrow \text{Int}(D)$.*

Let A be a ring and P an A -plethory. A P -ring is an A -algebra R together with an A -algebra homomorphism $\circ : P \odot_A R \rightarrow R$ such that $(\alpha \circ \beta) \circ r = \alpha \circ (\beta \circ r)$ and $e \circ r = e$ for all $\alpha, \beta \in P$ and all $r \in R$, where e is the image of X in the unit $A[X] \rightarrow P$ [2, 1.9]. Such a map \circ is said to be a *left action of P on R* . For example, P itself has a structure of a P -ring, as do the A -algebras $P \odot_A R$ and $\text{Hom}_A(P, R)$ for any A -algebra R [2, 1.10], with left actions given by

$$\begin{aligned} P \odot_A (P \odot_A R) &\longrightarrow P \odot_A R \\ \alpha \circ (\beta \circ r) &\longmapsto (\alpha \circ \beta) \circ r \end{aligned}$$

and

$$\begin{aligned} P \odot_A \text{Hom}_A(P, R) &\longrightarrow \text{Hom}_A(P, R) \\ \alpha \circ \varphi &\longmapsto (\beta \mapsto \varphi(\beta \circ \alpha)), \end{aligned}$$

respectively. Moreover, the functors $P \odot_A -$ and $W_P = \text{Hom}_A(P, -)$ from A -algebras to P -rings are left and right adjoints, respectively, for the forgetful functor from P -rings to A -algebras [2, 1.10].

For any A -plethory P , the P -ring $W_P(R) = \text{Hom}_A(P, R)$ of any A -algebra R is called the *P -Witt ring* of R . This terminology comes from the fact that, if P is the \mathbb{Z} -plethory Λ of [2, 2.11], then a P -ring is equivalently a λ -ring, and the functor W_P is isomorphic to the universal λ -ring functor Λ . If P is the \mathbb{Z} -plethory $\text{Int}(\mathbb{Z})$, then a P -ring is equivalently a binomial ring, and the functor W_P is isomorphic to the functor Bin . The latter fact generalizes to the following result, which implies Theorem 8 of the introduction.

Theorem 15. *Let D be an integral domain such that $\text{Int}(D)$ has a D -plethory structure with unit given by the inclusion $D[X] \rightarrow \text{Int}(D)$, and let A be a D -algebra.*

- (1) *If there exists an $\text{Int}(D)$ -ring structure on A , then A is WPC.*
- (2) *If A is D -torsion-free, then there exists a (necessarily unique) $\text{Int}(D)$ -ring structure on A if and only if A is WPC.*
- (3) *If A is D -torsion-free, then the D -algebra homomorphism $W_{\text{Int}(D)}(A) \rightarrow A$ is an inclusion with image equal to $w^D(A)$, and the functor w^D is therefore isomorphic to the functor $W_{\text{Int}(D)}$ restricted to the category of D -torsion-free D -algebras.*
- (4) *If A is D -torsion-free, then the D -algebra homomorphism $\text{Int}(D) \odot_D A \rightarrow A \otimes_D K$ acting by $f \odot a \mapsto f(a)$ has image equal to $w_D(A)$, and the functor w_D is therefore isomorphic to the functor $T \circ (\text{Int}(D) \odot_D -)$ restricted to the category of D -torsion-free D -algebras, where $T(B)$ for any D -algebra B denotes the image of B in $B \otimes_D K$, where K is the quotient field of D .*
- (5) *If A is D -torsion-free and WPC, then the natural D -algebra homomorphisms $W_{\text{Int}(D)}(A) \rightarrow A$ and $A \rightarrow T(\text{Int}(D) \odot_D A)$ are isomorphisms.*
- (6) *The functor $T \circ (\text{Int}(D) \odot_D -)$ is a left adjoint for the inclusion from D -torsion-free WPC D -algebras to D -algebras.*

- (7) The functors $W_{\text{Int}(D)}$ and $\text{Int}(D) \odot_D -$ are right and left adjoints, respectively, for the inclusion from D -torsion-free WPC D -algebras to D -algebras if and only if the $\text{Int}(D)$ -rings $W_{\text{Int}(D)}(R)$ and $\text{Int}(D) \odot_D R$ are D -torsion-free for every D -algebra R .
- (8) Every $\text{Int}(D)$ -ring is D -torsion-free if D is a PID with finite residue fields.

We end with the following problem.

Problem 16. Determine equivalent conditions on an integral domain D so that the D -algebra $\text{Int}(D)$ has a D -plethory structure with unit given by the inclusion $D[X] \rightarrow \text{Int}(D)$ and so that the D -algebras $W_{\text{Int}(D)}(R)$ and $\text{Int}(D) \odot_D R$ are D -torsion-free for every D -algebra R .

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