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Random Walks in Attractive Potentials: 
The Case of Critical Drifts

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Abstract

We consider random walks in attractive potentials - sub-additive functions of their local times. An application of a drift to such random walks leads to a phase transition: If the drift is small than the walk is still sub-ballistic, whereas the walk is ballistic if the drift is strong enough. The set of sub-critical drifts is convex with non-empty interior and can be described in terms of Lyapunov exponents (Sznitman, Zerner). Recently it was shown that super-critical drifts lead to a limiting speed. We shall explain that in dimensions $d \geq 2$ the transition is always of the first order. (Joint work with Y. Velenik)

1. Class of Models and Results

We consider nearest neighbour paths $\gamma = (\gamma(0), \ldots, \gamma(n))$ on $\mathbb{Z}^d$. The length of the path is denoted as $|\gamma| = n$ and its displacement is denoted as $X(\gamma) = \gamma(n) - \gamma(0)$. Unless mentioned otherwise all the paths start at the origin, $\gamma(0) = 0$.

Paths $\gamma$ are subject to a self-interacting potential $\Phi(\gamma)$ and to a drift $(h, X(\gamma)); h \in \mathbb{R}^d$. The potential $\Phi$ is of the form:

$$\Phi(\gamma) = \sum_{x \in \mathbb{Z}^d} \phi(\ell_\gamma(x)),$$

where $\ell_\gamma(x)$ is the local time of $\gamma$ at $x$. Here are our assumptions on $\phi$:

A1. $\phi(1) > 0$ and $\phi(\ell)$ is non-decreasing in $\ell$.

A2. $\phi(\ell + m) \leq \phi(\ell) + \phi(m)$.

A3. $\lim_{\ell \to \infty} \phi(\ell)/\ell = 0$ .

The assumption A2 means that $\Phi$ is a self-attractive potential. Assumption A3 is just a normalization. Assumption A1 ensures positivity of Lyapunov exponents (see below). The main example we have in mind is that of annealed random walks in random potentials, $\phi(\ell) = -\log \mathbb{E}e^{-\ell V}$, where $V$ is a non-negative random variable with $0 \in \text{supp}(V) \subseteq [0, \infty]$. Drifted Wiener sausage is a particular example. The $n$-step partition function is then given by

$$A_n^h = \sum_{|\gamma| = n} \left(\frac{1}{2d}\right)^{|\gamma|} e^{-\Phi(\gamma) + (h, X(\gamma))}.$$

Let $A_n^h$ to denote the corresponding path measure. There are two competing contributions to $A_n^h$: Because of the attractive nature of $\Phi$ paths prefer to collapse, whereas the drift $h$ pulls them away. The following is known [2,1]: Whichever $h$ one chooses, the mean displacement $X(\gamma)/n$ satisfies a large deviation principle under $A_n^h$ with a convex rate function $J^h$. Moreover, there exists a critical set of drifts $K_0$ - a compact convex subset of $\mathbb{R}^d$ with non-empty interior $0 \in \text{int}(K_0)$, such that:

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**Sub-ballistic drifts.** If $h \in \text{int}(K_0)$, then $J^h$ has a unique minimum at 0. In particular,

$$\lim_{n \to \infty} A_n^h \left( \frac{X(\gamma)}{n} \right) = 0.$$ 

**Ballistic drifts.** If $h \not\in K_0$ then $J^h$ has a unique minimum at some $v(h) \neq 0$. In particular,

$$\lim_{n \to \infty} A_n^h \left( \frac{X(\gamma)}{n} \right) = v(h).$$

**Critical drifts.** If $h \in \partial K_0$, then $J^h(0) = 0$.

Our main result implies that in any dimension $d \geq 2$ the transition is of the first order:

**Theorem A.** Let $h \in \partial K_0$. Then, there exists $v(h) \neq 0$, such that the rate function $J^h$ is zero on the segment $[0, v(h)]$ and strictly positive otherwise. Furthermore,

$$\lim_{n \to \infty} A_n^h \left( \frac{X(\gamma)}{n} \right) = v(h).$$

Actually our proof of this result implies accompanying laws under $A_n^h$:

**Theorem B.** The set of critical drifts is regular. Namely, $\partial K_0$ is locally analytic and has a uniformly positive Gaussian curvature. Let $h \in \partial K_0$ and let $v(h) \neq 0$ be as above. Then,

$$\lim_{n \to \infty} A_n^h \left( \frac{X(\gamma)}{n} - v(h) \right) > \epsilon = 0$$

for any $\epsilon > 0$. Moreover, there exists a non-degenerate covariance matrix $\Xi$, such that

$$\frac{X(\gamma) - nv(h)}{\sqrt{n}} \Rightarrow N(0, \Xi).$$

2. **Lyapunov Exponents**

The geometry of the problem is encoded in Lyapunov exponents: Given $x \in \mathbb{Z}^d$ and $\lambda \geq 0$ define

$$A_x^\lambda = \sum_{X(\gamma) = x} \left( \frac{1}{2d} \right)^{|\gamma|} e^{-\Phi(\gamma) - \lambda|\gamma|}.$$ 

Then,

$$a_\lambda(x) = -\lim_{N \to \infty} \frac{1}{N} \log A_x^{[Nx]}.$$ 

It is easy to check that the limit is well defined for any $x \in \mathbb{R}^d$ and $\lambda \geq 0$. Moreover $a_\lambda(\cdot)$ is an equivalent norm;

$$0 < \frac{1}{c_\lambda} \leq \min_{x \neq 0} \frac{a_\lambda(x)}{|x|} \leq \max_{x \neq 0} \frac{a_\lambda(x)}{|x|} \leq c_\lambda,$$

for any $\lambda \geq 0$.

The set of critical drifts is related to $a_0$ as follows:

$$K_0 = \{ h : (h, x) \leq a_0(x) \ \forall x \}.$$ 

Alternatively, one can describe $K_0$ as the closure of the domain of convergences of the series

$$h \mapsto \sum_{x \in \mathbb{Z}^d} e^{(h, x)} A_0^x.$$ 

For any $x \in \mathbb{R}^d$ one can choose $h \in \partial K_0$ such that

$$(h, x) = a_0(x) = \max_{g \in \partial K_0} (g, x).$$

In the sequel we shall fix a small number $\delta > 0$ and use it in order to quantify the cone of good directions $C_\delta(h)$ which is associated with a critical drift $h \in \partial K_0$. Namely,

$$C_\delta(h) = \left\{ x \in \mathbb{R}^d : (h, x) \geq (1 - \delta)a_0(x) \right\}.$$
3. Notes on the Proof

Let $h \in \partial K_0$, $x \in \mathbb{Z}^d$ and let $\gamma = (\gamma(0), \ldots, \gamma(k), \ldots, \gamma(m))$ be a path from 0 to $x$. We shall say that a point $u = \gamma(k)$ is an $h$-cone point of $\gamma$ if

$$\gamma \subseteq (u - C_\delta(h)) \cup (u + C_\delta(h)).$$

Here is the crucial result:

**The Mass Gap Estimate.** There exist $\delta, \eta, \nu > 0$ such that

$$e^{(h,x)} A^x_0 (\gamma \text{ has no } h\text{-cone points}) \leq e^{-\nu|x|}$$

uniformly in $h \in \partial K_0$ and in all $x \in C_\eta(h)$ sufficiently large.

The Mass-Gap estimates set up in motion the Ornstein-Zernike machinery developed in [1] and in references therein. An important new ingredient needed for the proof of the mass-gap is the following Lemma, which is used for controlling massless hairs of renormalized skeletons:

**Lemma.** Let $B_K$ be a Euclidean ball of radius $K$. Consider simple random walk paths $\gamma = (\gamma(0), \ldots, \gamma(\tau_K))$ which are run up to the first exit time from $B_K$. This gives rise to a probability distribution $Q_K$. For any path $\gamma$ as above define $R_K = R_K(\gamma)$ to be the size of its range (number of different points visited by $\gamma$ before $\tau_K$). Then for every $c_1 > 0$ there exists $c_2 > 0$ such that

$$Q_K (R_K \leq c_1 K) \leq e^{-c_2 K}$$

for all $K$ sufficiently large.

**References**
