Maki Furukado and Shunji Ito

Complex Pisot Numeration Systems

<http://acirm.cedram.org/item?id=ACIRM_2009__1__1_41_0>
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1. Background

Starting from an unimodular Pisot substitution, it is well-known that we can construct the numeration system. ([R], [A-I], [I-R], [F-F-I-W], etc.)

Let \( \sigma \) be \( \sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1 \) and the incidence matrix \( M_\sigma \) of \( \sigma \) is

\[
M_\sigma = \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

Then, \( A = M_\sigma^{-1} \) satisfies the complex Pisot condition, i.e. the eigenvalues of \( A \) satisfy

\[ \lambda (= \lambda_1), \lambda (= \lambda_2) \in \mathbb{C} \setminus \mathbb{R} \text{ and } |\lambda| = |X| > 1 > |\lambda_3|. \]

The family of compact sets \( \mathcal{P} = \{ \delta_1, \delta_2, \delta_3 \} \) of the \( A \)-invariant expanding plane \( P_e \), which is given by the projection method, i.e.

\[ \delta_i = \text{cl} \{ \{ w \sigma (s_1 s_2 \ldots s_{k-1}) | \exists k \in \mathbb{N}, s_k = i \} \}, \]

where \( w = s_1 s_2 \ldots = \lim_{n \to \infty} \sigma^n (1) \) is the fixed point of \( \sigma \), satisfies the following set equations:

\[ A \delta_1 = \delta_1 \cup \delta_2 \cup \delta_3, \quad A \delta_2 = \delta_1 + \pi_e e_3, \quad A \delta_3 = \delta_2 + \pi_e e_3. \]

Moreover, we know that we get the above family \( \mathcal{P} \) of compact sets by using the 2-dimensional extension \( E_2 (\theta) \) where \( \theta = \sigma^{-1} ([E]) \).

We then have the question: starting from an automorphism instead of a substitution, can we construct the numeration system?

For simplicity, we discuss the case of the complex Pisot number \( \lambda \) with degree 3 under some assumptions ([H-F-I]). The discussion in the case of degree 4 can be found in [A-F-H-I].

2. Notation

**Assumption 2.1.** (1) \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) is a complex Pisot number with degree 3, i.e. \( \lambda \) is the algebraic integer of the minimal polynomial:

\[ p_\lambda (x) = x^3 - ax^2 - bx \pm 1, \; a, b \in \mathbb{Z} \]

whose roots \( \lambda (= \lambda_1), \overline{\lambda} (= \lambda_2), \lambda_3 \) satisfy \( |\lambda| = |\overline{\lambda}| > 1 > |\lambda_3| \).

Text presented during the meeting “Numeration: mathematics and computer science” organized by Boris Adamczewski, Anne Siegel and Wolfgang Steiner. 23-27 mars 2009, C.I.R.M. (Luminy).

Key words. Quasi-periodic tiling, Rauzy fractal, Stepped plane.

This is the joint work with Masaki Hama.

41
(2) A is the 3×3 integer matrix such as $A_\pi := \begin{bmatrix} 0 & 0 & \mp 1 \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$ whose characteristic polynomial is given by $p_\pi(x)$ respectively.

Then, $A$ is called the companion Pisot matrix of $\lambda$.

Let $u_1$, $u_2$ be the eigenvectors of $A$ with respect to $\lambda_i$, $i = 1, 2$, and let $v_1$, $v_2$ be

$$v_1 := \frac{u_2 + u_1}{2}, \quad v_2 := \frac{u_2 - u_1}{2i}.$$ 

The $A$-expanding plane $P_c$ of $A$ is written by $P_c = \mathcal{L}(v_1, v_2)$ and the space $\mathbb{R}^3$ is decomposed by $P_c$ and the $A$-contractive line $P_c : \mathbb{R}^3 = P_c \oplus P_c$. Then, let us define the projection $\pi_x : \mathbb{R}^d \to P_c$ along $P_c$ by $\pi_x x = x_1$ for $x = x_1 + x_2 \in \mathbb{R}^d$ where $x_1 \in P_c$ and $x_2 \in P_c$.

**Lemma 2.2.** Put $\lambda = a + bi$, $a, b \in \mathbb{R}$, then $A[v_1v_2] = [v_1v_2] \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

**Definition 2.3** (Complex Pisot numeration system). For a complex Pisot number $\lambda$, if we can find the finite family of compact sets $\mathcal{P} = \{\gamma_j\}_{j \in \{1, 2, 3\}}$ of $P_c$ with the finite integer vector sequence $\{f_k^{(j)}\}_{1 \leq k \leq l_j}$, $f_k^{(j)} \in \mathbb{Z}^d$ and the finite index sequence $\{V_k^{(j)}\}_{1 \leq k \leq l_j}$, $V_k^{(j)} \in \{1, 2, 3\}$ satisfying

(N1) $\mu_e(\gamma_j) > 0$, $\operatorname{cl}(\mu_e(\pi_{\gamma_j})) = \gamma_j$, and $\mu_e(\partial \gamma_j) = 0$

where $\mu_e$ is the Lebesgue measure on $P_c$, $\operatorname{int}(A)$ is the interior of the set $A$, $\operatorname{cl}(A)$ is the closure of the set $A$, and $\partial \gamma_j := \gamma_j \backslash \operatorname{int}(\gamma_j)$;

(N2) for each $j \in \{1, 2, 3\}$, the set equation holds:

$$A\gamma_j = \bigcup_{k=1}^{l_j} \left( \gamma^{(j)}_{k} + \pi_v f_k^{(j)} \right) \quad \text{(disjoint)}$$

where $\pi_v : \mathbb{R}^d \to P_c$ is the projection along $P_c$, $\bigcup_k A_k$ (disjoint)’ means that $\operatorname{int}(A_k) \cap \operatorname{int}(A_{k'}) = \emptyset$ if $k \neq k'$;

(N3) $\bigcup_{j \in \{1, 2, 3\}} \gamma_j$ (disjoint),

then, we say that the pair $(A, \mathcal{P})$ is the complex Pisot numeration system of $\lambda$.

The reason why we call $(X, \mathcal{P})$ the complex Pisot numeration system is that $x \in \bigcup_{j=1, 2, 3} X_j$ is written as

$$x = \sum_{n=1}^{\infty} A^{-n} \left( \pi_v f_{k_n}^{(j_{n-1})} \right)$$

by the integer vector sequence $\left( f_{k_0}^{(j_0)}, f_{k_1}^{(j_1)}, \ldots, f_{k_{n-1}}^{(j_{n-1})}, \ldots \right)$

satisfying $x = x_0 \in \gamma_0$, $x_n = A x_{n-1} - \pi_v f_{k_{n-1}}^{(j_{n-1})}$.

**Question 1.** When a complex Pisot number $\lambda$ is given, how do we obtain the complex Pisot numeration system? In other words, how do we obtain the finite family of compact sets $\mathcal{P} = \{\gamma_j\}_{j=1, 2, 3}$ of $P_c$ satisfying (N1), (N2), (N3)?

3. Results

First, we classify the distribution of $\lambda_i$, $i = 1, 2, 3$ of $A$ into the following four types:

<table>
<thead>
<tr>
<th>$\lambda_3$</th>
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<tbody>
<tr>
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</tr>
<tr>
<td>$\Rightarrow 27 - 4a^3 - 18ab - 4b^2 - 4b^3 &gt; 0$.</td>
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</tbody>
</table>

$\Re(\lambda_1) = 0$ $\Re(\lambda_1) > 0$ $\Re(\lambda_1) = 0$ $\Re(\lambda_1) < 0$

<table>
<thead>
<tr>
<th>$a \geq 0$</th>
<th>$a &lt; 0$</th>
<th>$a &gt; 0$</th>
<th>$a \leq 0$</th>
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Let $v^*$ and $v$ be the left and right-eigenvectors of $\lambda_3$: $v^* A = \lambda_3 v^*$, $A v = \lambda_3 v$, then, we see that $v^* = [1, \lambda_3, \lambda_3^2]$, $v = [\frac{1}{\lambda_3}, \lambda_3 - a, 1]$.

**Lemma 3.1.** The expanding plane $P_e$ can be characterized by using $v^*$:

$$P_e = \{ x = (x_1, x_2, x_3) \mid \langle x, v^* \rangle = 0 \}$$

where $(y, z)$ means the inner product of vectors $x$ and $y$.

From the signature of $v^*$, we have the following:

<table>
<thead>
<tr>
<th>sgn $v^*$</th>
<th>type 1</th>
<th>type 2</th>
<th>type 3</th>
<th>type 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-, +, +]$</td>
<td>$[+, -,+]$</td>
<td>$[+, +,+]$</td>
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</table>

![Diagram of types 1, 2, 3, 4](image)

For $x \in \mathbb{R}^3$, $i, j \in \{\pm 1, \pm 2, \pm 3\}$, let $(x, i \wedge j)$ be the 2-dimensional positive oriented unit face $i \wedge j$ located at $x \in \mathbb{Z}^3$, i.e.

$$(x, i \wedge j) := \{ x + \lambda (\text{sgn}(i)) e_{i|} + \mu (\text{sgn}(j)) e_{j|} \mid x \in \mathbb{Z}^3, 0 \leq \lambda, \mu \leq 1 \}.$$

<table>
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<tr>
<th>sgn $v$</th>
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<th>type 3</th>
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![Diagram of unit faces V1, V2, V3, V4](image)

By using the 2-dimensional positive oriented unit faces $V_t$, $t = 1, 2, 3, 4$, we define

- the set of unit faces $S_t$ of $P_e$: $S_t := \{ (x, i \wedge j) \mid x \in \mathbb{Z}^3, \{i, j,k\} = \{1, 2, 3\}, i \wedge j \in V_t, \langle x, v^* \rangle \geq 0, \langle x - e_k, v^* \rangle < 0 \}$;
- the family of finite sets of unit faces $G_t$: $G_t := \{ \sum_{\mu \in \Lambda} (x, i \wedge j)_\mu \mid \# \Lambda < +\infty, (x, i \wedge j)_\mu \in S_t \}$;
- the stepped plane $\mathcal{S}_t$ of $P_e$: $\mathcal{S}_t := \bigcup_{(x, i \wedge j) \in S_t} (x, i \wedge j)$.

The stepped plane $\mathcal{S}_t$ is the surface generated by 2-dimensional positive oriented unit faces. We know that the projection $\pi^* : \mathcal{S}_t \rightarrow P_e$ along $v^*$ is bijective and there are not any lattice points except $0$ between $P_e$ and $\mathcal{S}_t$.

For the companion matrix $A_-$ ($A_+$), let us choose the automorphism $\theta_-$ ($\theta_+$) on the free group $F(1, 2, 3)$ whose incidence matrix is $A_-$ ($A_+$) respectively as follows:

\[
\begin{align*}
\theta_- : & \quad 1 \rightarrow 2 \quad 1 \rightarrow 2 \\
& \quad 2 \rightarrow 3 \quad 2 \rightarrow 3 \\
& \quad 3 \rightarrow 3^{a_1} \cdot 2^b \quad 3 \rightarrow 12^b \cdot 3^a.
\end{align*}
\]
Using the automorphism \( \theta \), let us define the 2-dimensional extension \( E_2 (\theta) : G_t \rightarrow G_t \) as follows:

\[
E_2 (\theta) (0, i \wedge j) := (0, \theta (i) \wedge \theta (j)) := \sum_{1 \leq k \leq l_i} \left( f \left( P_k^{(i)} \right) + f \left( P_k^{(j)} \right) \right)
\]

\[
E_2 (\theta) (x, i \wedge j) := Ax + E_2 (\theta) (0, i \wedge j)
\]

\[
E_2 (\theta) \left( \sum_{\mu} (x, i \wedge j)_{\mu} \right) := \sum_{\mu} \left( E_2 (\theta) (x, i \wedge j)_{\mu} \right)
\]

where \( f : F (1, 2, 3) \rightarrow \mathbb{Z}^3 \) is the homomorphism satisfying

\[
f (0) = 0, \ f (i) = e_i, \ \theta (i) = W_1^{(i)} W_2^{(i)} \ldots W_k^{(i)},
\]

\( P_k^{(i)} \) is the prefix of \( W_k^{(i)} \), i.e., \( P_k^{(i)} = W_1^{(i)} W_2^{(i)} \ldots W_{k-1}^{(i)} \), and \( y + (0, i \wedge j) = (y, i \wedge j) \). We will show the behavior of \( E_2 (\theta) \) on some examples.

3.1. The results on type 1 and type 4. Let us consider the case \((a, b) = (1, 0)\) of type 1 as an example, i.e. \( A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \) and its characteristic polynomial is \( x^3 - x^2 + 1 \). The distribution of the eigenvalues of \( A \) is \( \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \). Then the automorphism \( \theta : 1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 31^{-1} \) is determined and \( E_2 (\theta) \) is given by

\[
E_2 (\theta) (0, 1 \wedge 2) = (0, \theta (1) \wedge \theta (2)) = (0, 2 \wedge 3)
\]

\[
E_2 (\theta) (0, 1 \wedge 3) = (0, 2 \wedge 3 \wedge 1^{-1}) = (0, 2 \wedge 3) + (e_3, 2 \wedge 1^{-1}) \stackrel{(\ast)}{=} (0, 2 \wedge 3) + ((e_3 - e_1), 1 \wedge 2)
\]

\[
E_2 (\theta) (0, 2 \wedge 3) = (0, 3 \wedge 3 \wedge 1^{-1}) \stackrel{(\ast)}{=} ((e_3 - e_1), 1 \wedge 3)
\]

where \((\ast)\) means the rearrangement.

We can see that \( E_2 (\theta) \) generates the patches whose orientations are positive in general on type 1 and we obtain the following propositions on type 1.

Proposition 3.2. On type 1, we consider

\[
\mathcal{U}_1 := (x - e_2, 1 \wedge 2) + (x, 1 \wedge 3) + (x + e_1, 2 \wedge 3)
\]

where \( x \) is the solution of \( x + e_3 - e_2 = Ax + ae_3 - e_1 \). Then, \( E_2 (\theta)^2 (\mathcal{U}_1) \supset \mathcal{U}_1 \).
The figure of $E_2(\theta)^n(U_1)$, $n = 0, 1, 2, \ldots, 7$ in the case of $(a, b) = (1, 0)$ of type 1.

Let us define

\[ T_1 := \left\{ \pi_e(x, i \wedge j) \mid (x, i \wedge j) \in E_2(\theta)^{2n}U_1(x) \right\}, \]

\[ \gamma_{i\wedge j} := \lim_{n \to \infty} A^{-n} \pi_eE_2(\theta)^n(x_{i\wedge j}, i \wedge j) \text{ for } (x_{i\wedge j}, i \wedge j) \in U_1(x). \]

Then,

1. $T_1$ is the quasi-periodic polygonal tiling of $P_e$ and $T_1 = T_1^\circ - \pi_e x$
   where $T_1^\circ := \{\pi_e(x, i \wedge j) \mid (x, i \wedge j) \in S_1\};$
2. $P_1 := \{\gamma_{i\wedge j} \mid i \wedge j \in V_1\},$ then $(A, P_1)$ is the complex Pisot numeration system;
3. $\hat{T}_1 := \{\pi_e x + \gamma_{i\wedge j} \mid \pi_e(x, i \wedge j) \in T_1\}$ is a self-similar tiling of $P_e$.

Analogous results are obtained on type 4.

**Proposition 3.3.** On type 4, we consider

\[ U_4 := (0, 1 \wedge 2) + (0, 3 \wedge 1) + (0, 2 \wedge 3). \]

Then, $E_2(\theta)(U_4) \succeq U_4$. Let us define

\[ T_4 := \{\pi_e(x, i \wedge j) \mid (x, i \wedge j) \in E_2(\theta)^nU_4\}, \]

\[ \gamma_{i\wedge j} := \lim_{n \to \infty} A^{-n} \pi_eE_2(\theta)^n(x_{i\wedge j}, i \wedge j) \text{ for } (x_{i\wedge j}, i \wedge j) \in U_4. \]

Then,

1. $T_4$ is the quasi-periodic polygonal tiling of $P_e$ and $T_4 = T_4^\circ$
   where $T_4^\circ := \{\pi_e(x, i \wedge j) \mid (x, i \wedge j) \in S_4\};$
2. $P_4 := \{\gamma_{i\wedge j} \mid i \wedge j \in V_4\},$ then $(A, P_4)$ is the complex Pisot numeration system;
3. $\hat{T}_4 := \{\pi_e x + \gamma_{i\wedge j} \mid \pi_e(x, i \wedge j) \in T_4\}$ is a self-similar tiling of $P_e$.

**Remark 3.4.** We see that the numeration system produced from the unimodular Pisot substitution.

3.2. **The results on type 2.** Let us observe the case $(a, b) = (-1, -2)$ of type 2 as an example,
i.e. $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$ and its characteristic polynomial is $x^3 + x^2 + 2x + 1$. The eigenvalues of $A$
is distributed as $\begin{array}{ccc} 1 & 2 & 3 \\ 1 \quad 2 \quad 1 & 3 & 1 \end{array}$, then the automorphism $\theta : 2 \mapsto 3 \quad 1 \mapsto 1^{-1}2^{-1}1^{-1}3$ is determined and

$E_2(\theta)$ is given by

\[
E_2(\theta)(0, 1 \wedge 2) := (0, 2 \wedge 3) \\
E_2(\theta)(0, 3 \wedge 1) := (-e_1, 2 \wedge 1) + (-e_1 - 2e_2 - e_3, 2 \wedge 3) \\
E_2(\theta)(0, 2 \wedge 3) := (-e_1, 1 \wedge 3) + (-e_1 - e_2, 2 \wedge 3) + (-e_1 - 2e_2, 2 \wedge 3)
\]
In the case of type 2, $E_2(\theta)$ generates the patches including the negative faces. How shall we treat the negative faces?

Let us introduce the blocking method on type 2 and we see that the blocking method works well on type 2 in general.

Proposition 3.5. On type 2, let us consider $U_2$:

$$U_2 := \text{block (1)} + \text{block (2)} + \text{block (3)}$$

where

$$\text{block (1)} := (x + e_2, 1 \land 3) + (x, 2 \land 3) + \sum_{k=1}^{a-1} (x - ke_2, 2 \land 3);$$
$$\text{block (2)} := (x - e_2, 2 \land 3);$$
$$\text{block (3)} := (x, 1 \land 2);$$

and $x$ is the solution of $x = A^2x - e_1 - e_2$. Then, $E_2(\theta)$ satisfies $E_2(\theta)^2U_2 \succ U_2$.

Moreover,

1. $C_2 := \{\pi_\varepsilon(y + \text{block (i)}) \mid (y + \text{block (i)}) \in E_2(\theta)^n(\text{block (j)}), \text{block (j)} \in U_2, \text{for some } j \in \{1, 2, 3\}\}$ is the covering of $P_\varepsilon$;
2. $\gamma_1 := \lim_{n \to \infty} A^{-2n}\pi_\varepsilon E_2(\theta)^{2n}(\text{block (i)})$, then $(A, P_2, P_2 = \{\gamma_1, \gamma_2, \gamma_3\}$ is the complex Pisot numeration system;
3. $\tilde{C}_2 := \{\pi_\varepsilon y + \gamma_1 \mid \pi_\varepsilon (y + \text{block (i)}) \in C_2\}$ is a self-similar tiling of $P_\varepsilon$. 

46
3.3. The results on type 3. Let us give an example in the case of \((a, b) = (1, -2)\) of type 3.

\[
A = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & -2 \\
0 & 1 & 1
\end{bmatrix}
\]

and its characteristic polynomial is \(x^3 - x^2 + 2x - 1\). The distribution of the eigenvalues of \(A\) is \([-1.5, 1.5]\). Then the automorphism \(\theta : 1 \mapsto 2 \mapsto 3 \mapsto 312^{-1}2^{-1}\) is determined and \(E_2(\theta)\) is given by

\[
\begin{align*}
E_2(\theta)(0, 1 \land 2) & := (0, 2 \land 3) \\
E_2(\theta)(0, 1 \land 3) & := (0, 2 \land 3) + (e_3, 2 \land 1) \\
E_2(\theta)(0, 2 \land 3) & := (e_3, 3 \land 1) + (e_1 - e_2 + e_3, 2 \land 3) + (e_1 - 2e_2 + e_3, 2 \land 3)
\end{align*}
\]

In this case, \(E_2(\theta)\) also generates patches including the negative unit faces. On this example, the blocking method doesn’t seem to work well. But on the stepped plane, we can find \(\theta'\) with P. Arnoux by the private discussion whose incidence matrix is same as the companion matrix and we succeed in finding the tiling substitution \(\theta^* : G_3 \rightarrow G_3, \theta^* \neq E_2(\theta)\). However, it is unclear whether this method can apply all of type 3 in general.
REFERENCES


Yokohama
Kanazawa